CS 273A: Machine Learning

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Lecture 3: Bayes Classifiers

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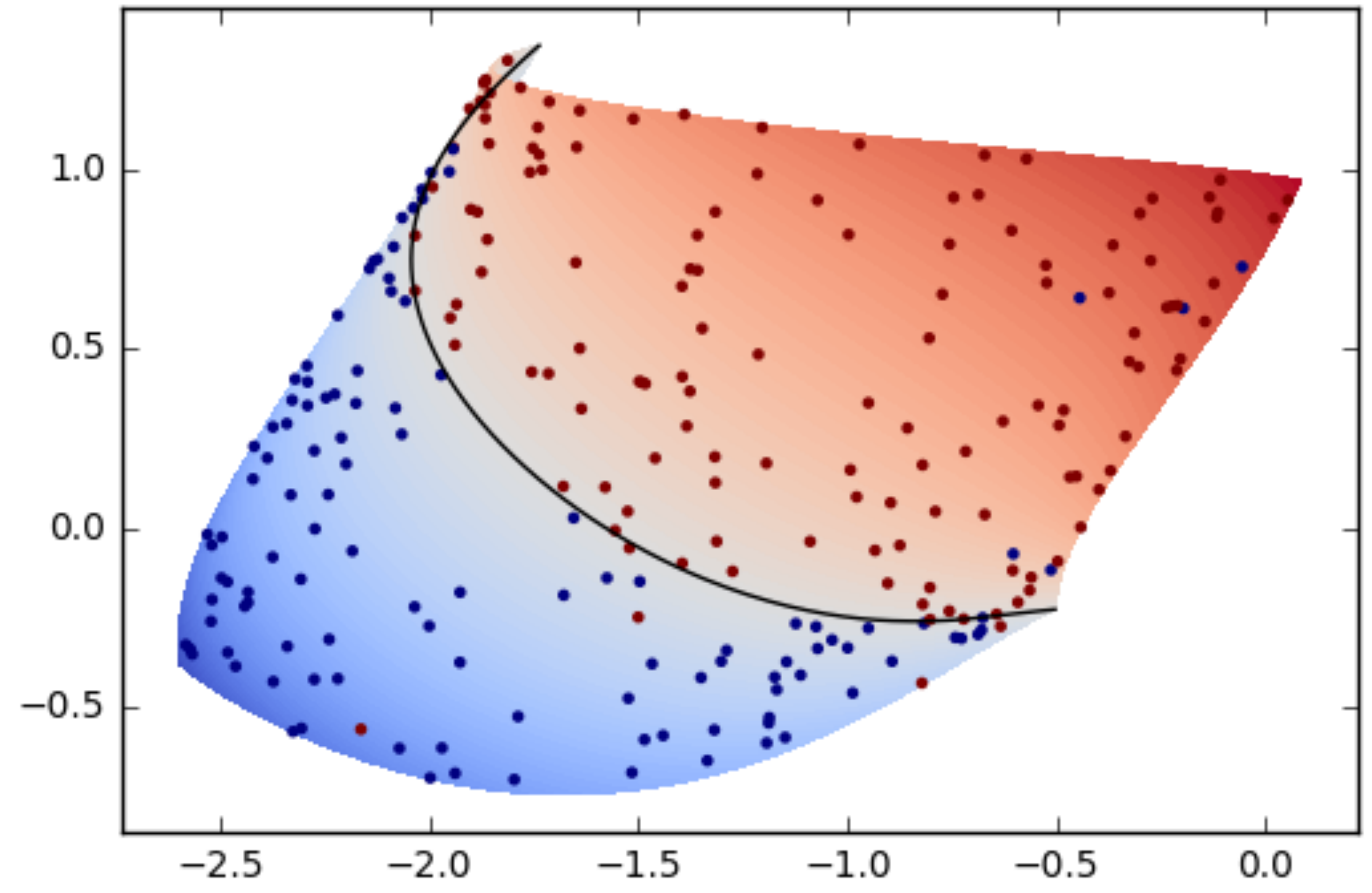
All slides in this course adapted from Alex Ihler & Sameer Singh



Logistics

assignment 1

- Assignment 1 is due Thursday

resources

- A list of great ML textbooks is on the website

Today's lecture

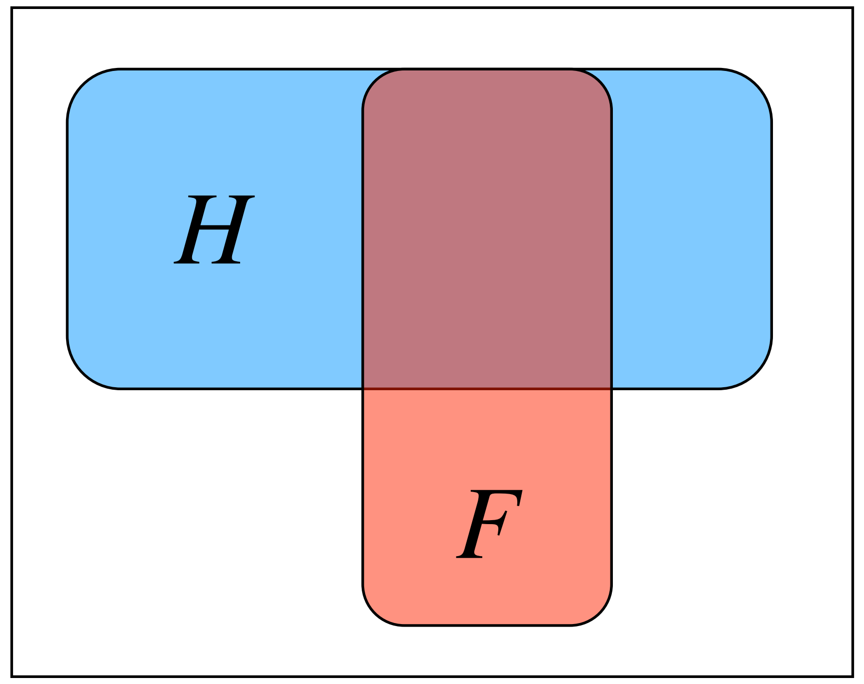
Bayes classifiers

Naïve Bayes Classifiers

Bayes error

Conditional probabilities

- Two events: headache (H), flu (F)
- $p(H) = \frac{1}{10}$
- $p(F) = \frac{1}{40}$
- $p(H \mid F) = \frac{1}{2}$
- You wake up with a headache
  - What are the chances that you have the flu?



$$p(F, H) = p(F)p(H \mid F)$$
$$= \frac{1}{40} \cdot \frac{1}{2} = \frac{1}{80}$$

$$p(F \mid H) = \frac{p(F, H)}{p(H)}$$
$$= \frac{1}{80} \cdot \frac{10}{1} = \frac{1}{8}$$

Example from Andrew Moore's slides

Probabilistic modeling of data

- Assume data with features x and discrete labels y
- Prior probability of each class: $p(y)$
  - Prior = before seeing the features
  - E.g., fraction of applicants that have good credit
- Distribution of features given the class: $p(x \mid y = c)$
  - How likely are we to see x in applicants with good credit?
- Joint distribution: $p(x, y) = p(x)p(y \mid x) = p(y)p(x \mid y)$
- Bayes' rule: posterior $p(y \mid x) = \dfrac{p(y)p(x \mid y)}{p(x)} = \dfrac{p(y)p(x \mid y)}{\sum_c p(y = c)p(x \mid y = c)}$

models:

$x \longrightarrow y$

$y \longrightarrow x$

does not imply causality!

Bayes classifiers

- Learn a "class-conditional" model for the data
  - Estimate the probability for each class $p(y = c)$
  - Split training data by class $\mathcal{D}_c = \{x^{(j)} : y^{(j)} = c\}$
  - Estimate from $\mathcal{D}_c$ the conditional distribution $p(x \mid y = c)$
- For discrete x, can represent as a contingency table

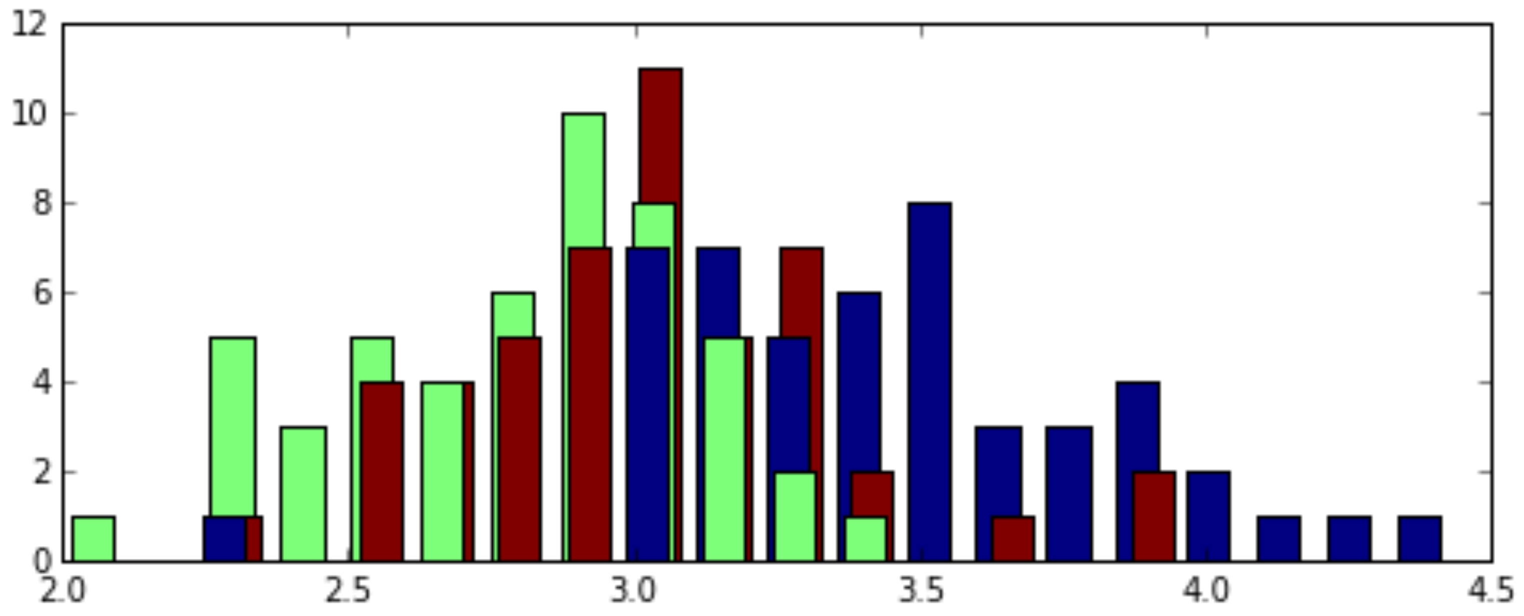
Features	# bad	# good
X=0	42	15
X=1	338	287
X=2	3	5
p(y)	**383/690**	**307/690**

p(x|y=0)	p(x|y=1)
42/383	15/307
338/383	287/307
3/383	5/307

p(y=0|x)	p(y=1|x)
.7368	.2632
.5408	.4592
.3750	.6250

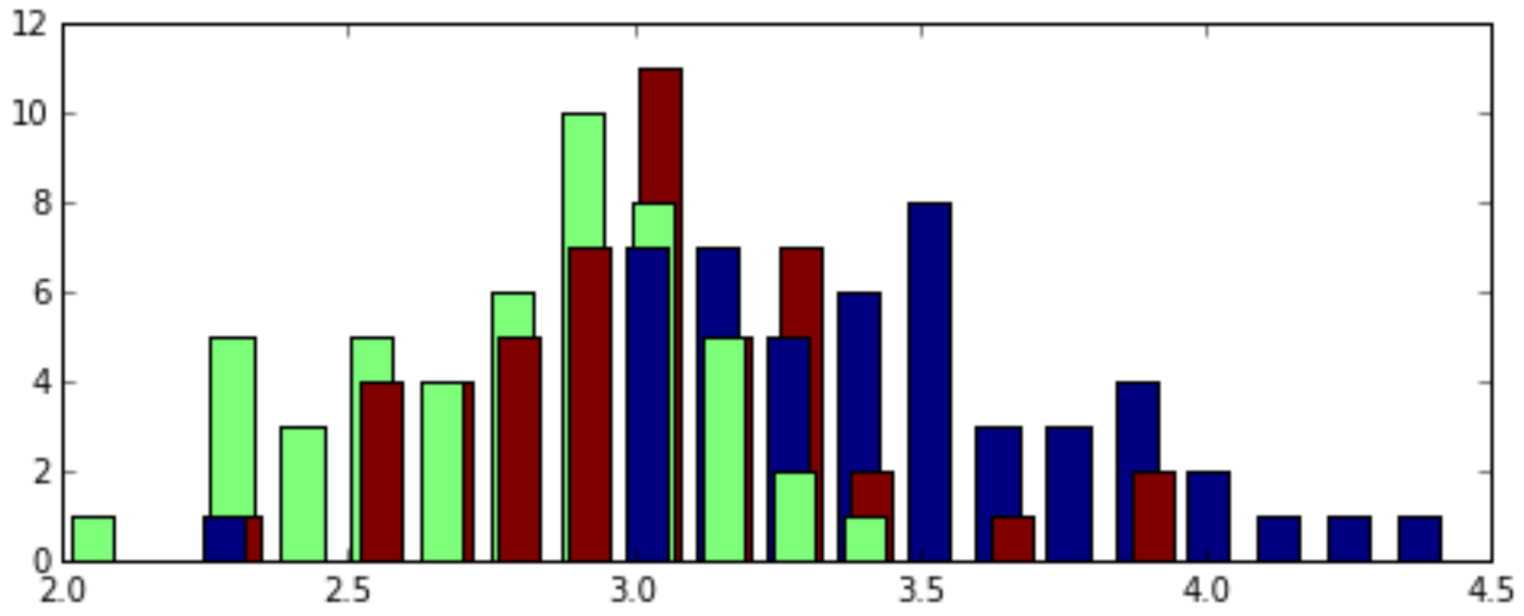
Bayes classifiers

- Learn a “class-conditional” model for the data
  - Estimate the probability for each class $p(y = c)$
  - Split training data by class $\mathscr{D}_c = \{x^{(j)} : y^{(j)} = c\}$
  - Estimate from $\mathscr{D}_c$ the conditional distribution $p(x \mid y = c)$
- For continuous x, we need some other density model
  - Histogram
  - Gaussian
  - others...



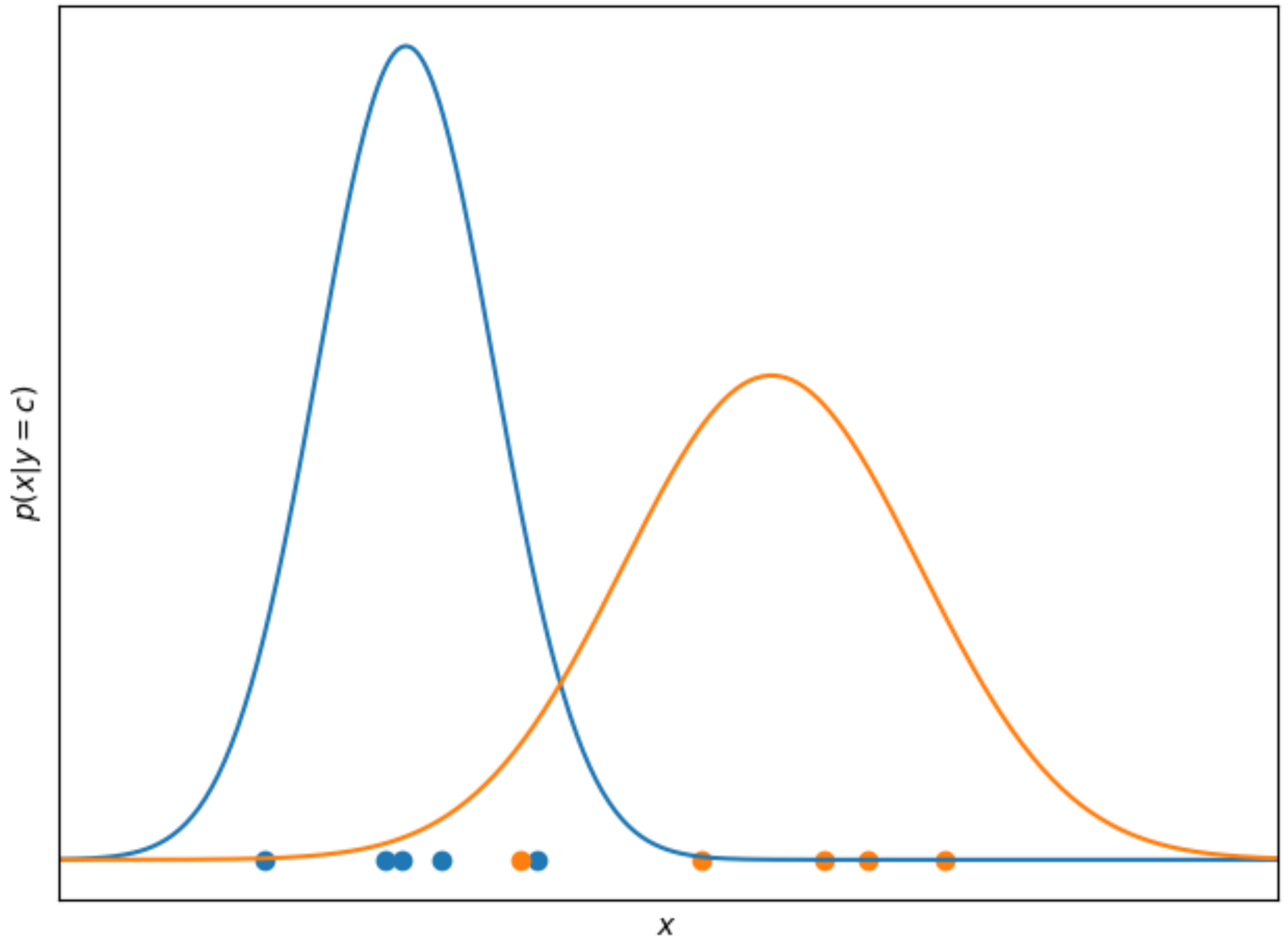
Histograms

- Split training data by class $\mathcal{D}_c = \{x^{(j)} : y^{(j)} = c\}$
- For each class, split x into k bins and count data points in each bin
- Normalize the k-dimensional count vector to get $p(x \mid y = c)$
- To use: given x, find its bin, output probability for that bin



Gaussian models

- Model instances in each class with a Gaussian $p(x \mid y = c) \sim \mathcal{N}(\mu_c, \sigma_c^2)$
- Estimate parameters of each Gaussians from the data $\mathcal{D}_c$
  - $\hat{p}(y = c) = \dfrac{m_c}{m}$ where $m_c = |\mathcal{D}_c|$
  - $\hat{\mu}_c = \dfrac{1}{m_c} \sum_{j:\ y^{(j)}=c} x^{(j)}$
  - $\hat{\sigma}_c^2 = \dfrac{1}{m_c} \sum_{j:\ y^{(j)}=c} (x^{(j)} - \hat{\mu}_c)^2$



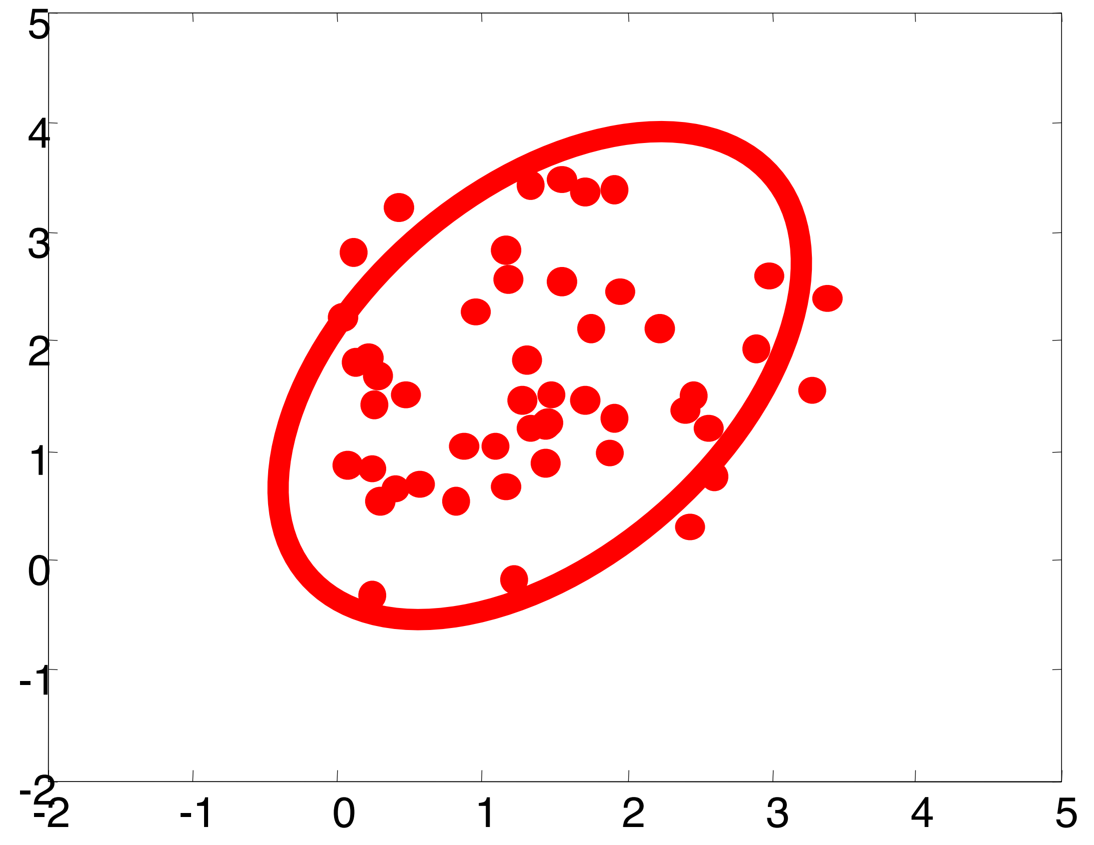
Multivariate Gaussian models

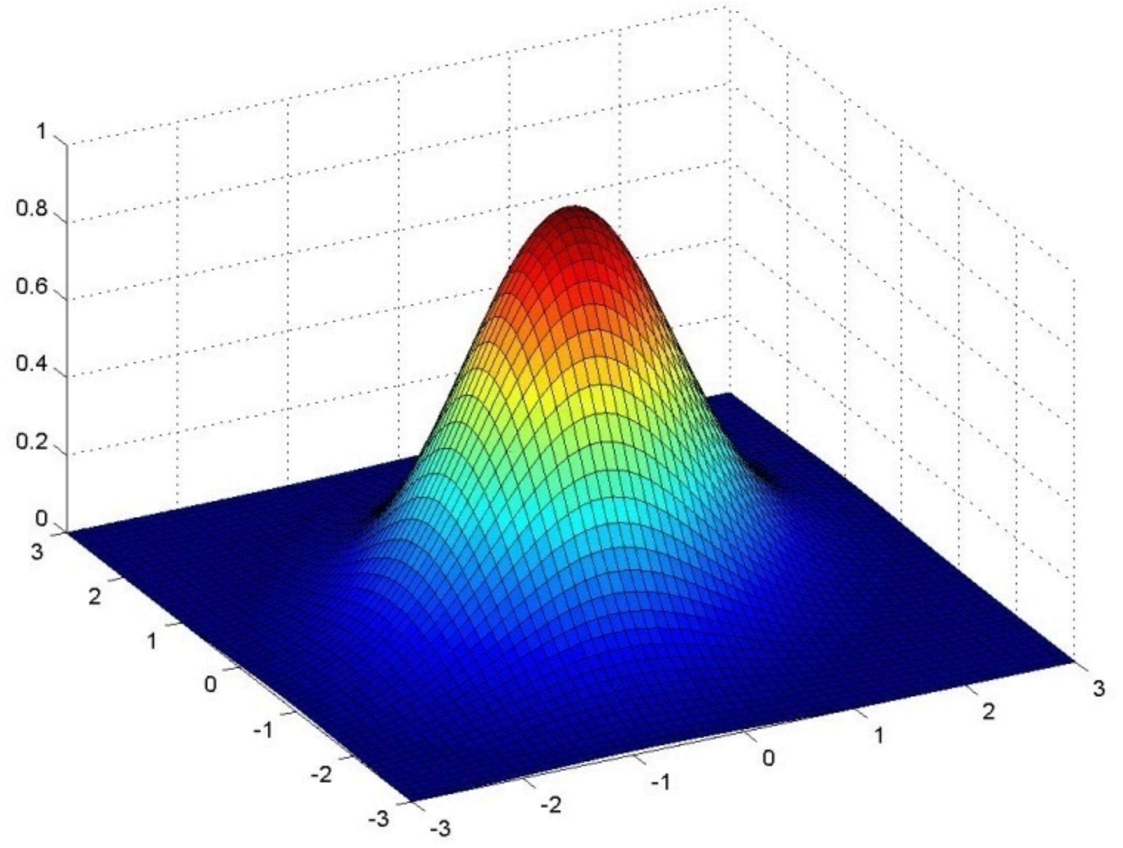
- Multivariate Gaussian: $\mathcal{N}(x; \mu, \Sigma) = (2\pi)^{-\frac{d}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^\intercal \Sigma^{-1}(x-\mu)\right)$

- Estimation similar to univariate case:
  - $\hat{\mu}_c = \frac{1}{m_c} \sum_j x^{(j)}$
  - $\hat{\Sigma}_c = \frac{1}{m_c} \sum_j (x^{(j)} - \hat{\mu}_c)(x^{(j)} - \hat{\mu}_c)^\intercal$ (outer product)

μ = mean (d-dimensional vector)
Σ = covariance ($d \times d$ matrix)
Σ^{-1} = precision ($d \times d$ matrix)
$|\cdot|$ = determinant (scalar)

- How many parameters?
  - $d + d^2$

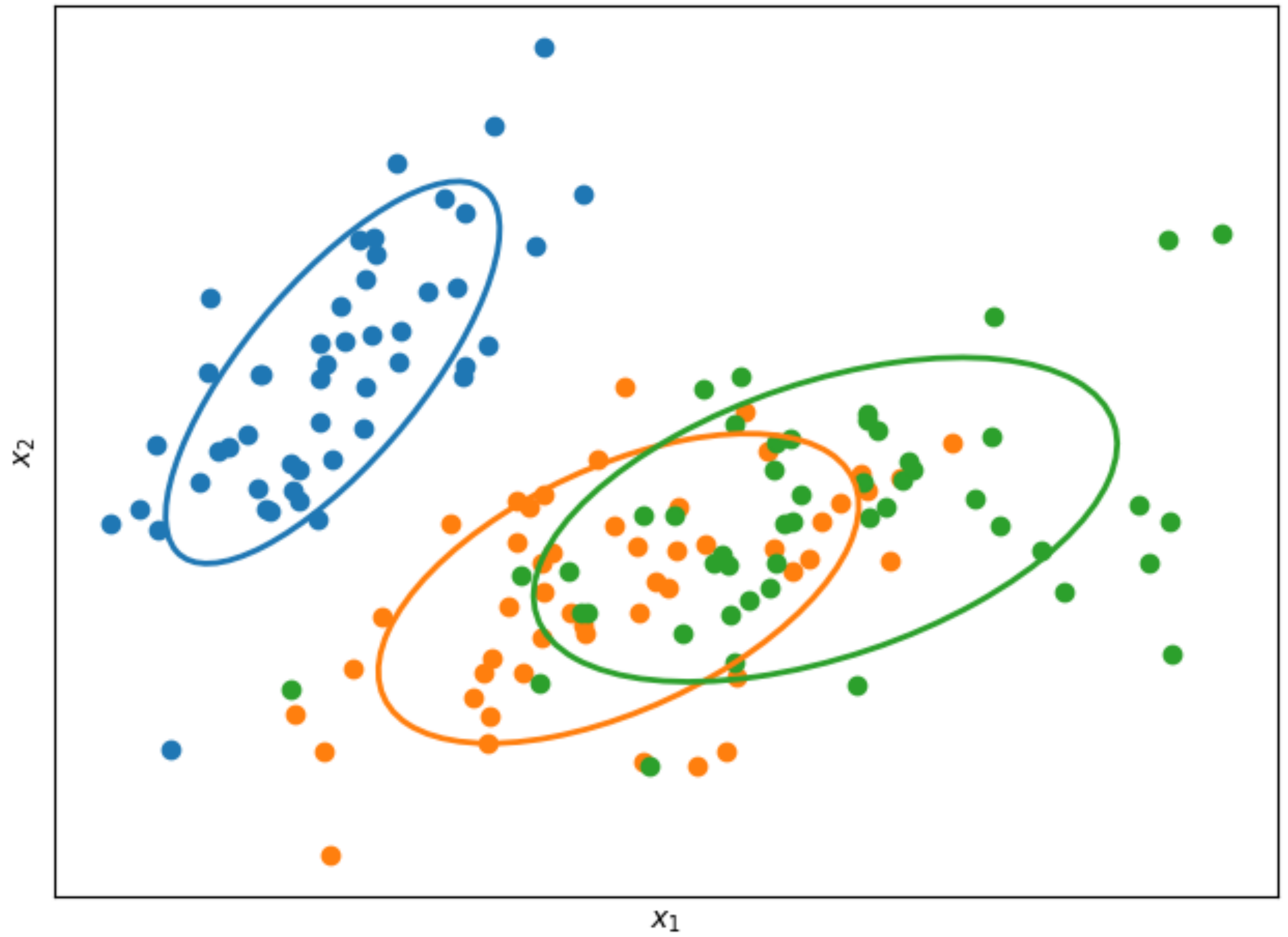




Gaussian Bayes: Iris example

- $\hat{p}(y=c) = \frac{50}{150}; y \sim \text{Categorical}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

- Fit mean and covariance for each class, $\hat{p}(x \mid y=c) = \mathcal{N}(x; \hat{\mu}_c, \hat{\Sigma}_c)$



- How to use:
  - $\hat{p}(y \mid x) = \frac{\hat{p}(y)\hat{p}(x \mid y)}{\hat{p}(x)} \propto \hat{p}(y)\hat{p}(x \mid y)$
  - Maximum posterior (MAP): $\hat{y}(x) = \arg\max_y \hat{p}(y)\hat{p}(x \mid y)$

Today's lecture

Bayes classifiers

Naïve Bayes Classifiers

Bayes error

Representing joint distributions

- Assume data with binary features
- How to represent $p(x \mid y)$?
- Create a truth table of all x values

A	B	C
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Representing joint distributions

- Assume data with binary features
- How to represent $p(x \mid y)$?
- Create a truth table of all x values
- Specify $p(x \mid y)$ for each cell
- How many parameters?
  - $2^n - 1$

A	B	C	p(A,B,C | y=1)
0	0	0	0.50
0	0	1	0.05
0	1	0	0.01
0	1	1	0.10
1	0	0	0.04
1	0	1	0.15
1	1	0	0.05
1	1	1	0.10

Estimating joint distributions

- Can we estimate $p(x \mid y)$ from data?
- Count how many data points for each x?
  - If $m \ll 2^n$, most instances never occur
    - Do we predict that missing instances are impossible?
      - What if they occur in test data?
- Difficulty to represent and estimate go hand in hand
  - Model complexity → overfitting!

A	B	C	p(A,B,C | y=1)
0	0	0	4/10
0	0	1	1/10
0	1	0	0/10
0	1	1	0/10
1	0	0	1/10
1	0	1	2/10
1	1	0	1/10
1	1	1	1/10

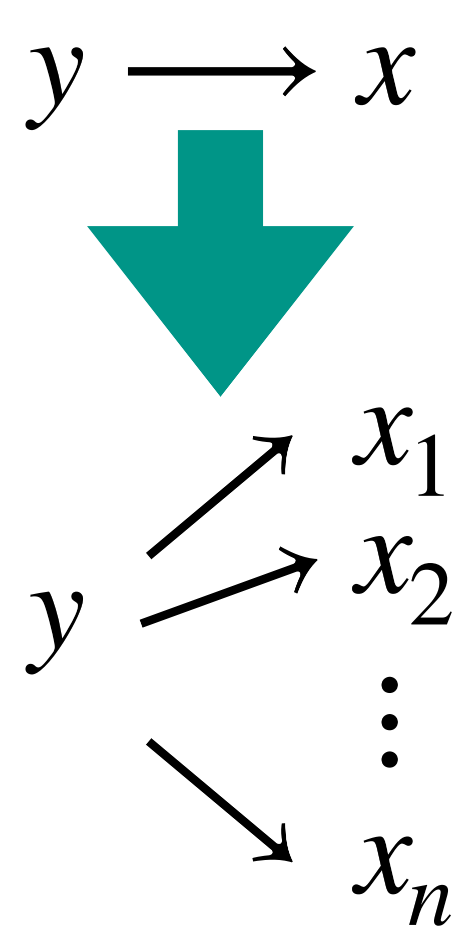
Regularization

- Reduce effective size of model class
  - Hope to avoid overfitting
- One way: make the model more “regular”, less sensitive to data quirks
- Example: add small “pseudo-count” to the counts (before normalizing)
  - $\hat{p}(x \mid y = c) = \dfrac{\#_c(x) + \alpha}{m_c + \alpha \cdot 2^n}$
  - Not a huge help here, most cells will be uninformative $\dfrac{\alpha}{m_c + \alpha \cdot 2^n}$

Simplifying the model

- Another way: reduce model complexity

- Example: assume features are independent of one another (in each class)
  - $p(x_1, x_2, \ldots, x_n \mid y) = p(x_1 \mid y) p(x_2 \mid y) \cdots p(x_n \mid y)$



- Now we only need to represent / estimate each $p(x_i \mid y)$ individually

A	p(A |y=1)
0	.4
1	.6

B	p(B |y=1)
0	.7
1	.3

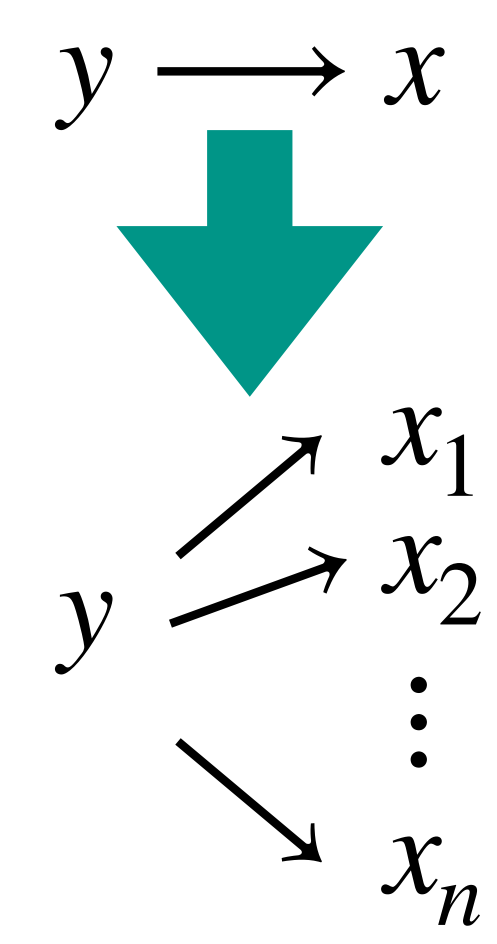
C	p(C |y=1)
0	.1
1	.9

$\Rightarrow$

A	B	C	p(A,B,C | y=1)
0	0	0	.4 * .7 * .1
0	0	1	.4 * .7 * .9
0	1	0	.4 * .3 * .1
0	1	1	…
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Naïve Bayes models

- We want to predict some value y, e.g. auto accident next year
- We have many known indicators for y (covariates) $x = x_1, \ldots, x_n$
  - E.g., age, income, education, zip code, ...
  - Learn $p(y \mid x_1, \ldots, x_n)$ — but cannot represent / estimate $O(2^n)$ values
- Naïve Bayes
  - Estimate prior distribution $\hat{p}(y)$
  - Assume $p(x_1, \ldots, x_n \mid y) = \prod_i p(x_i \mid y)$, estimate covariates independently $\hat{p}(x_i \mid y)$
  - Model: $\hat{p}(y \mid x) \propto \hat{p}(y) \prod_i \hat{p}(x_i \mid y)$



causal structure wrong!
(but useful...)

Naïve Bayes models: example

- $y \in \{\text{spam, not spam}\}$
- x = observed words in email
  - E.g., [“the” ... “probabilistic” ... “lottery” ...]
  - $x = [0,1,0,0,\ldots,0,1]$ (1 = word appears; 0 = otherwise)
- Representing $p(x \mid y)$ directly would require $2^{\text{thousands}}$ parameters
- Represent each word indicator as independent (given class)
  - Reducing model complexity to thousands of parameters
- Words more likely in spam pull towards higher $p(\text{spam} \mid x)$, and v.v.

Numeric example

- $\hat{p}(y=1) = \frac{4}{8} \quad = 1 - \hat{p}(y=0)$

- $\hat{p}(x_1, x_2 \mid y) = \hat{p}(x_1 \mid y)\hat{p}(x_2 \mid y)$

- $\hat{p}(x_1 = 1 \mid y = 0) = \frac{3}{4} \qquad \hat{p}(x_1 = 1 \mid y = 1) = \frac{2}{4}$

- $\hat{p}(x_2 = 1 \mid y = 0) = \frac{2}{4} \qquad \hat{p}(x_2 = 1 \mid y = 1) = \frac{1}{4}$

x_1	x_2	y
1	1	0
1	0	0
1	0	1
0	0	0
0	1	1
1	1	0
0	0	1
1	0	1

- What to predict for $x_1, x_2 = 1,1$? **prediction:** $\hat{y} = 0$
  - $\hat{p}(y=0)\hat{p}(x=1,1 \mid y=0) = \frac{4}{8} \cdot \frac{3}{4} \cdot \frac{2}{4} \qquad \hat{p}(y=1)\hat{p}(x=1,1 \mid y=1) = \frac{4}{8} \cdot \frac{2}{4} \cdot \frac{1}{4}$

Numeric example

- $\hat{p}(y=1) = \frac{4}{8} \quad = 1 - \hat{p}(y=0)$
- $\hat{p}(x_1, x_2 | y) = \hat{p}(x_1 | y)\hat{p}(x_2 | y)$
- $\hat{p}(x_1 = 1 | y = 0) = \frac{3}{4} \qquad \hat{p}(x_1 = 1 | y = 1) = \frac{2}{4}$
- $\hat{p}(x_2 = 1 | y = 0) = \frac{2}{4} \qquad \hat{p}(x_2 = 1 | y = 1) = \frac{1}{4}$
- What is $\hat{p}(y = 1 | x_1 = 1, x_2 = 1)$?

x_1	x_2	y
1	1	0
1	0	0
1	0	1
0	0	0
0	1	1
1	1	0
0	0	1
1	0	1

$$\frac{\hat{p}(y=1)\hat{p}(x=1,1|y=1)}{\hat{p}(x=1,1)} = \frac{\hat{p}(y=1)\hat{p}(x=1,1|y=1)}{\hat{p}(y=0)\hat{p}(x=1,1|y=0) + \hat{p}(y=1)\hat{p}(x=1,1|y=1)} = \frac{\frac{4}{8}\cdot\frac{2}{4}\cdot\frac{1}{4}}{\frac{4}{8}\cdot\frac{3}{4}\cdot\frac{2}{4} + \frac{4}{8}\cdot\frac{2}{4}\cdot\frac{1}{4}} = \frac{1}{4}$$

Recap

- Bayes' rule: $p(y \mid x) = \dfrac{p(y)p(x \mid y)}{p(x)}$
- Bayes classifiers: estimate $p(y)$ and $p(x \mid y)$ from data
- Naïve Bayes classifiers: assume independent features $p(x \mid y) = \prod_i p(x_i \mid y)$
  - Estimate each $p(x_i \mid y)$ individually
- Maximum posterior (MAP): $\hat{y}(x) = \arg\max_y p(y \mid x) = \arg\max_y p(y)p(x \mid y)$
  - Normalizer $p(x)$ not needed

Today's lecture

Bayes classifiers

Naïve Bayes Classifiers

Bayes error

Bayes classification error

- What is the training error of the MAP prediction $\hat{y}(x) = \arg\max_y p(y \mid x)$?

Features	# bad	# good	prediction:
X=0	42	15	bad
X=1	338	287	bad
X=2	3	5	good

errors

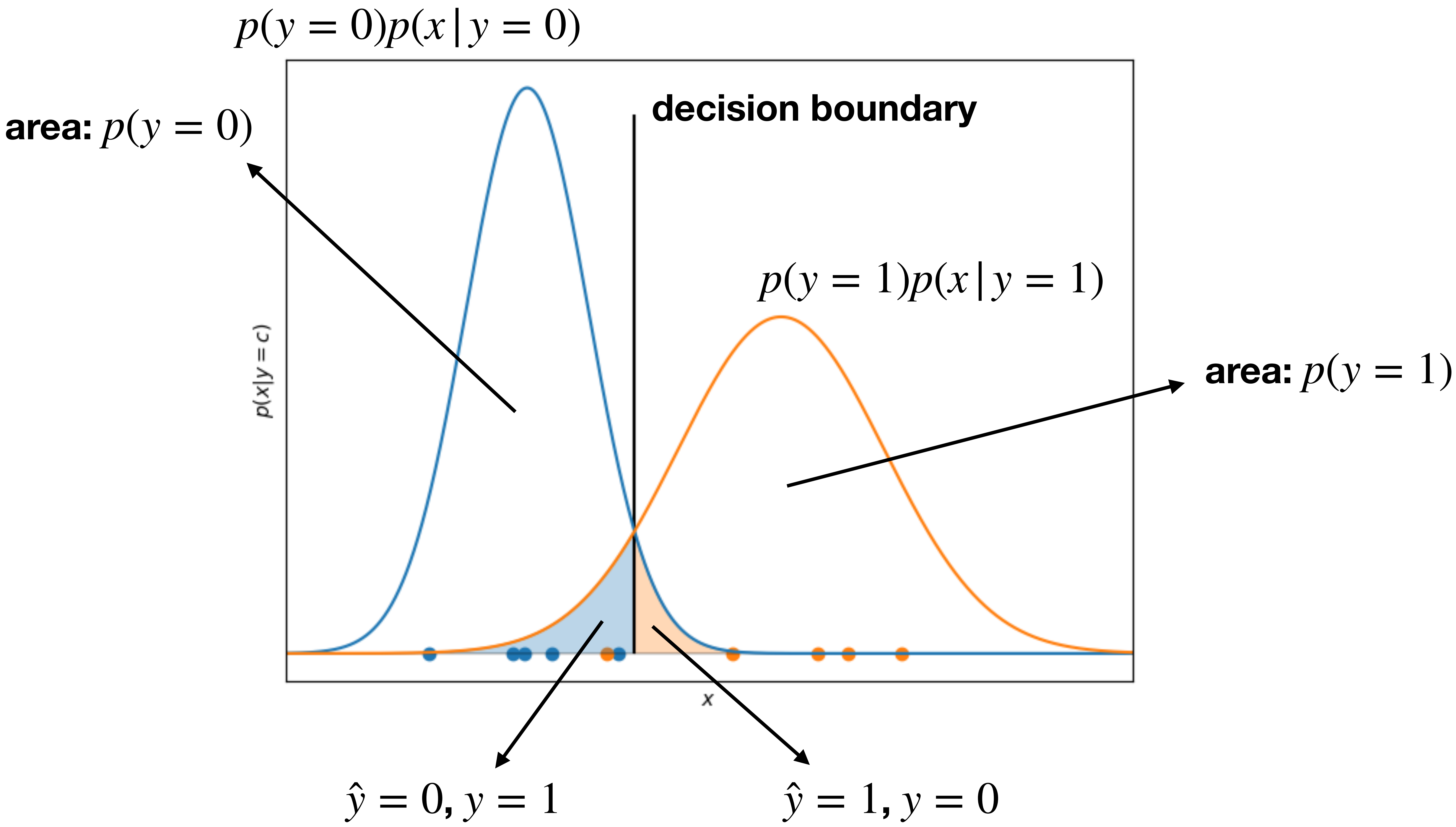
- $p(\hat{y} \neq y) = \dfrac{15 + 287 + 3}{690} = 0.442$

- Bayes error rate: probability of misclassification by MAP of true posterior

Bayes error rate

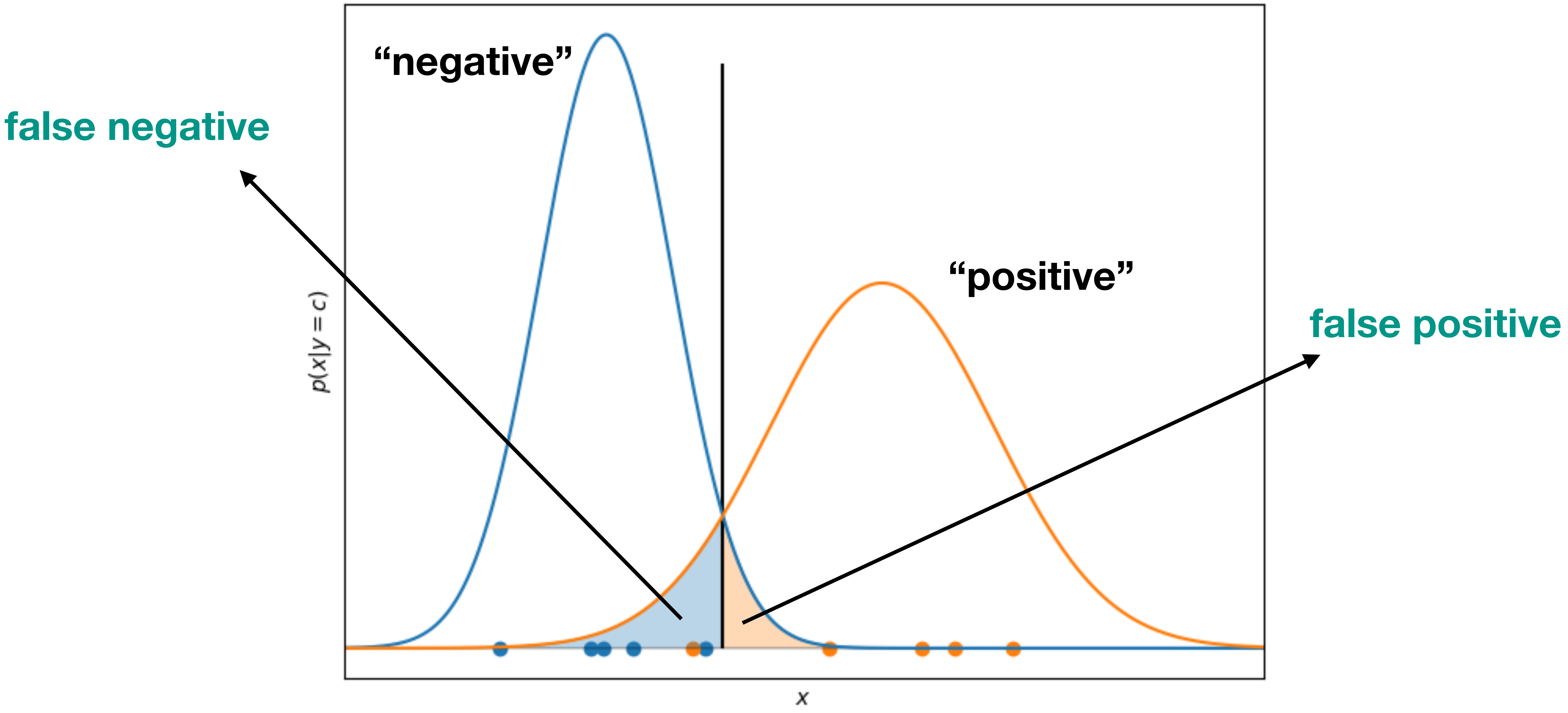
- Suppose that we know the true probabilities $p(x, y)$
  - And that we can compute prior $p(y)$ and posterior $p(y|x)$
- Bayes-optimal decision = MAP: $\hat{y} = \arg\max_y p(y|x)$
- Bayes error rate: $\mathbb{E}_{x,y\sim p}[\hat{y} \neq y] = \mathbb{E}_{x\sim p}[1 - \max_y p(y|x)]$
  - This is the optimal error rate of any classifier
  - Measures intrinsic hardness of separating y values given only x
    - But may get better with more features
- Normally we cannot estimate the Bayes error rate, only approximate with good classifier

Bayes error rate: Gaussian example



Types of error

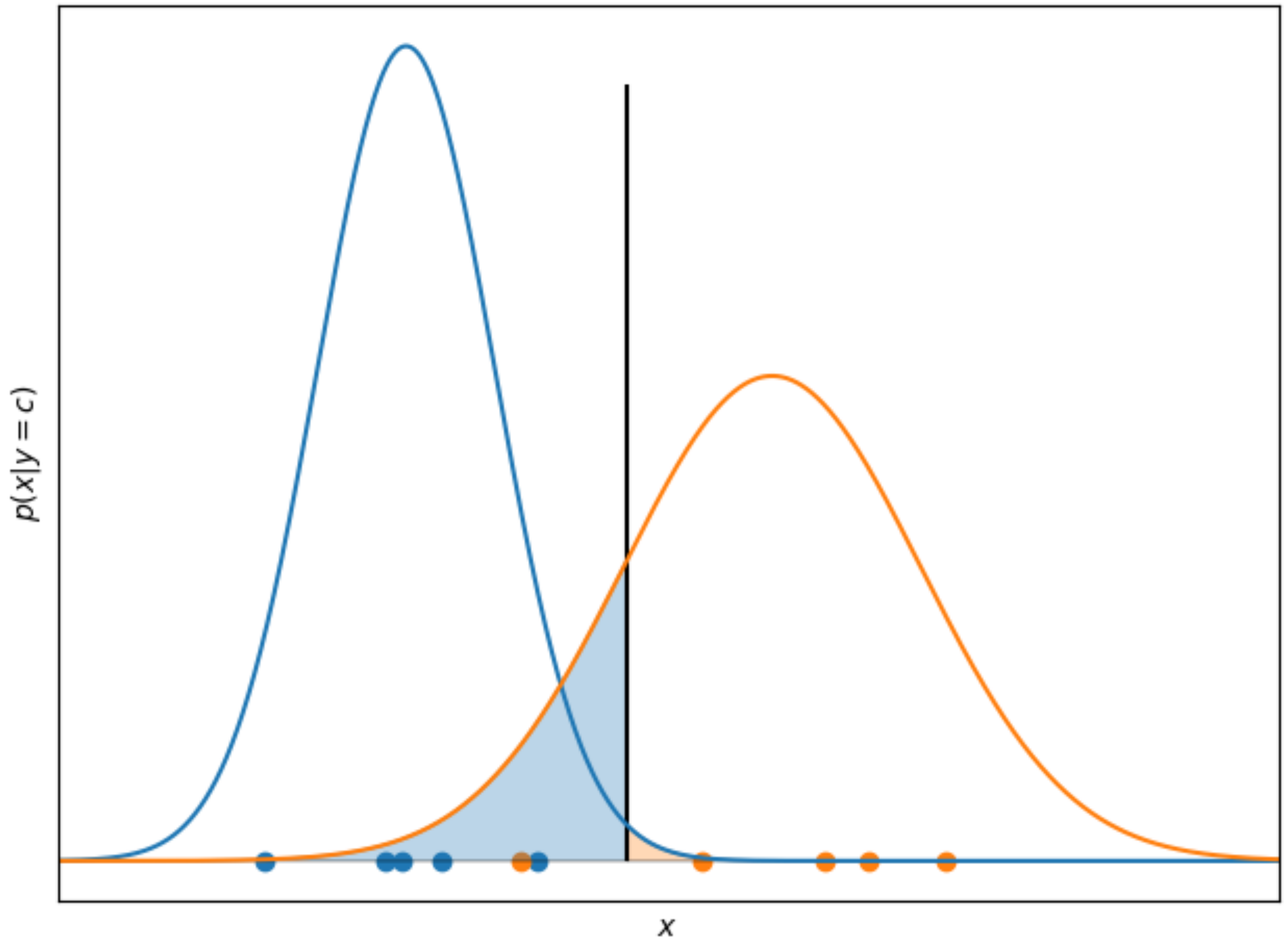
- Not all errors are equally bad
  - Do some cost more? (e.g. red / green light, diseased / healthy)



- False negative rate: $\dfrac{p(y=1,\hat{y}=0)}{p(y=1)}$; false positive rate: $\dfrac{p(y=0,\hat{y}=1)}{p(y=0)}$

Cost of error

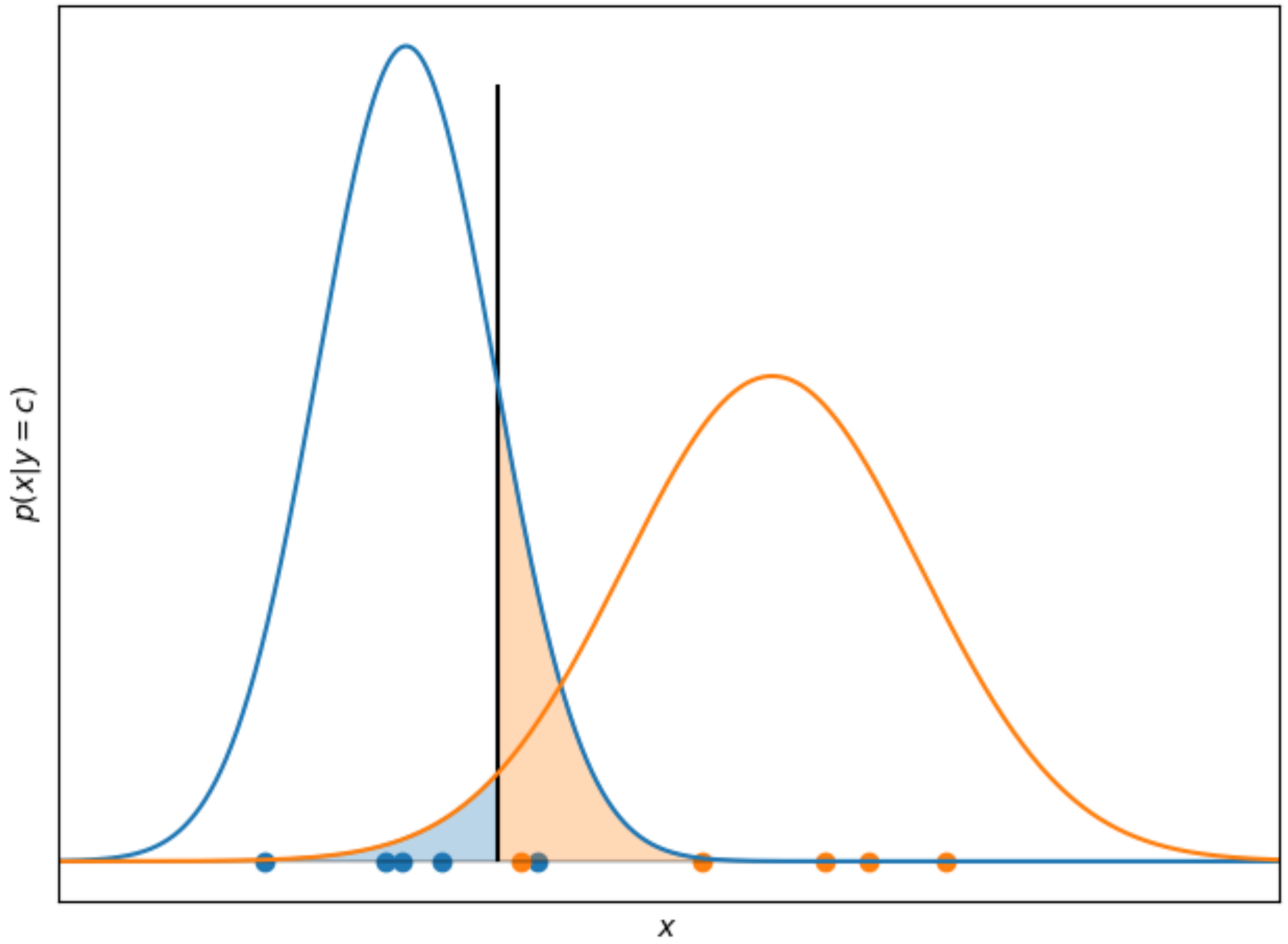
- Weight different costs differently
  - $\alpha \cdot p(y=0)p(x \mid y=0) \lessgtr p(y=1)p(x \mid y=1)$



- Increase α to prefer class 0

Cost of error

- Weight different costs differently
  - $\alpha \cdot p(y=0)p(x\,|\,y=0) \lessgtr p(y=1)p(x\,|\,y=1)$



- Decrease α to prefer class 1

Logistics

assignment 1

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resources

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