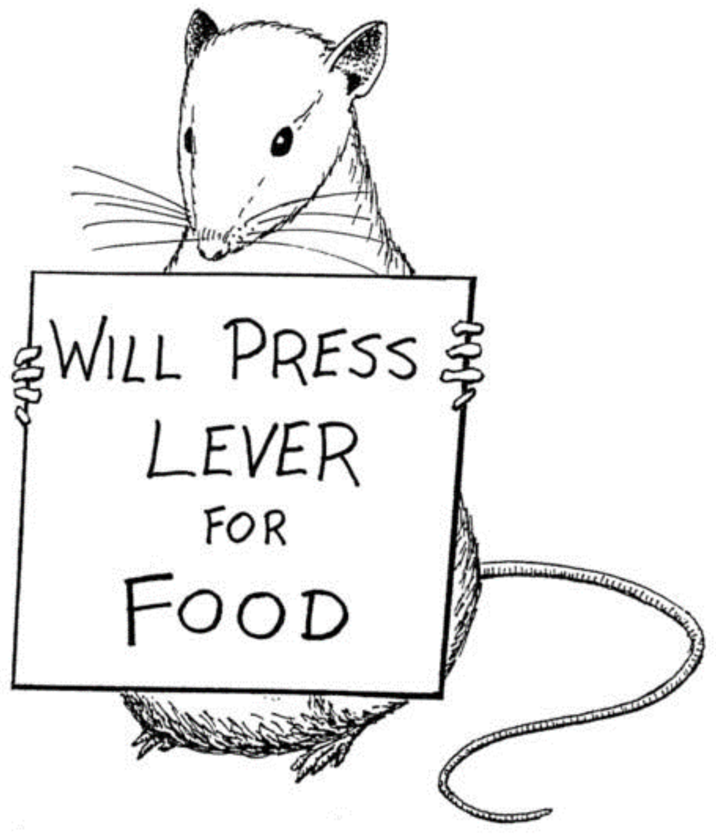
CS 277: Control and Reinforcement Learning

Winter 2022

Lecture 8: Optimal Control

Roy Fox
Department of Computer Science
Bren School of Information and Computer Sciences
University of California, Irvine



Today's lecture

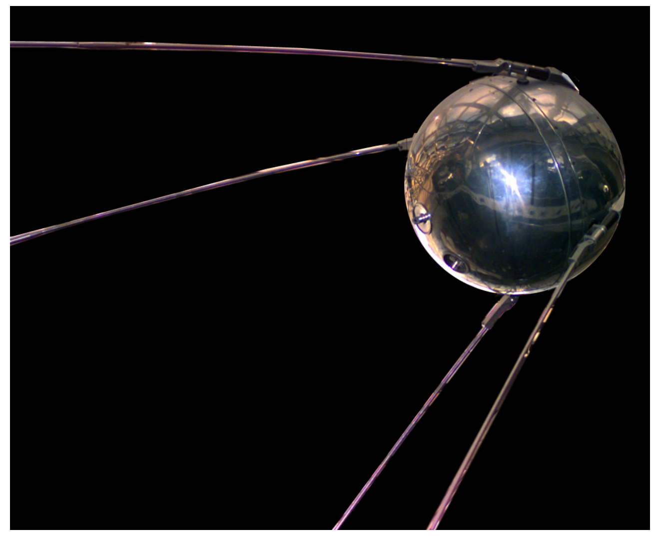
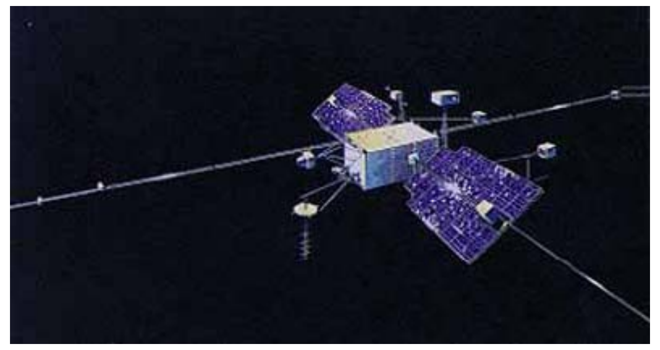
Stability, reachability, stabilizability

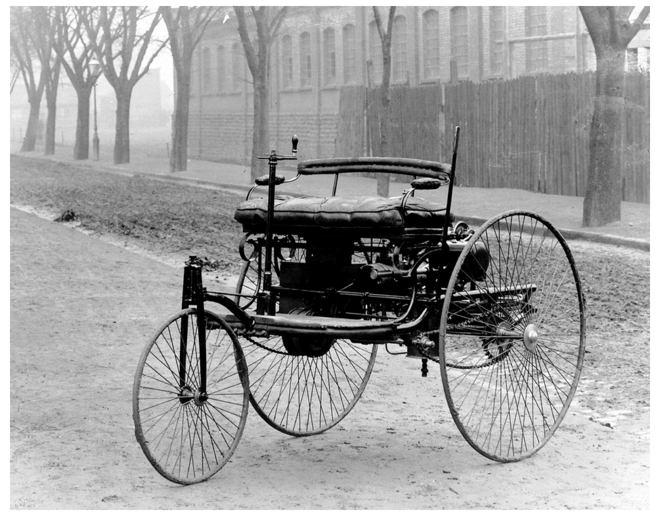
Linear Quadratic Regulator

Hamiltonian

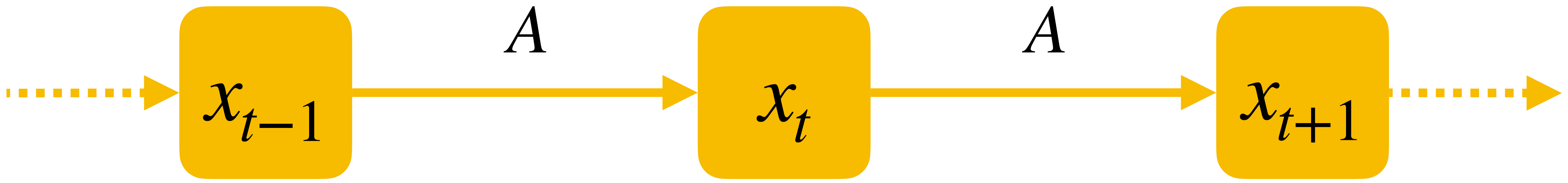
Why Optimal Control?

- Optimal Control involves environments simple enough to solve directly
  - Important applications
  - Powerful and profound theory
  - Useful insights / components for harder domains


Linear Time-Invariant (LTI) systems



- Continuous state space: $x_t \in \mathbb{R}^n$
- Simplest system — linear: $x_{t+1} = Ax_t \qquad A \in \mathbb{R}^{n \times n}$
  - Linear Time-Invariant (LTI): A does not depend on t
- How does the system evolve over time?

$$x_t = A^t x_0$$

- Adding drift b doesn't add insight

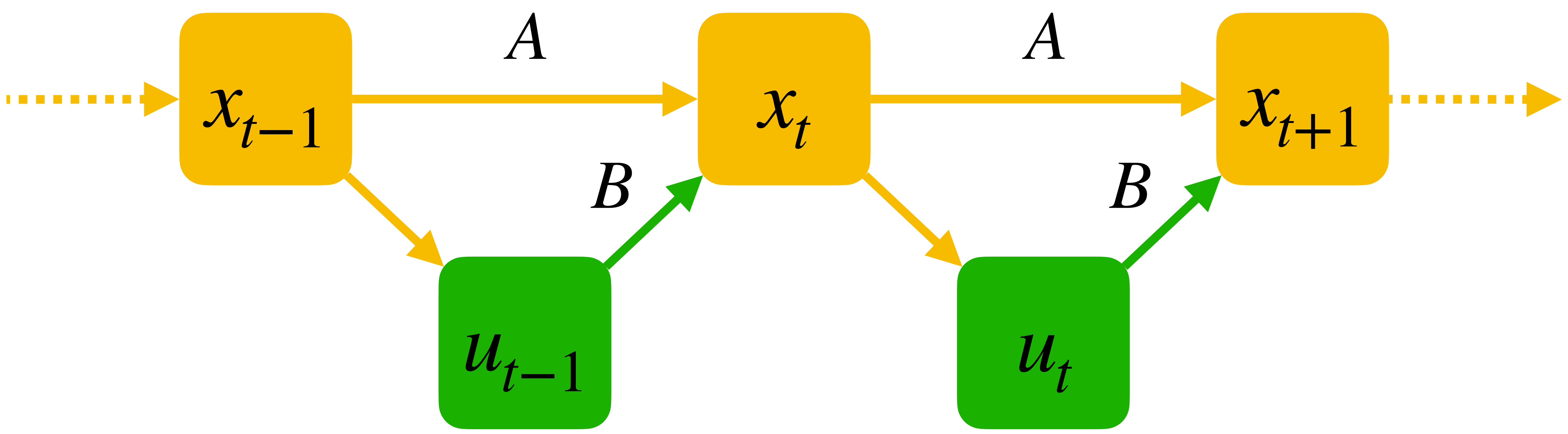
Stability

- To analyze: use eigenvectors $\lambda e = Ae$
- Consider a basis of eigenvectors $e_1, \ldots, e_n \in \mathbb{C}^n$

$$x_0 = \sum_i \alpha_i e_i \implies x_1 = Ax_0 = \sum_i \alpha_i \lambda_i e_i \implies x_t = \sum_i \alpha_i \lambda_i^t e_i$$

- Instability: some $\|\lambda_i\| > 1$, so that $\lim_{t\to\infty} \|x_t\| \to \infty$
- Stability: all $\|\lambda_i\| < 1$, so that $\lim_{t\to\infty} x_t = 0$
  - When $\|\lambda_i\| = 1$, component never vanishes or explodes; still called unstable

Linear control systems



- Continuous action (control) space: $u_t \in \mathbb{R}^m$
- Controlled LTI system: $x_{t+1} = Ax_t + Bu_t \qquad B \in \mathbb{R}^{n\times m}$

$$x_t = A^t x_0 + A^{t-1}Bu_0 + \cdots + ABu_{t-2} + Bu_{t-1}$$

$$x_t = A^t x_0 + \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix} \begin{bmatrix} u_{t-1} \\ u_{t-2} \\ \vdots \\ u_0 \end{bmatrix}$$

Reachability

$$x_t = A^t x_0 + \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix} \begin{bmatrix} u_{t-1} \\ u_{t-2} \\ \vdots \\ u_0 \end{bmatrix}$$

- Can we reach a given state x_t at time t?
  - If and only if $x_t - A^t x_0 \in \operatorname{span} \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix}$
- Cayley-Hamilton: A satisfies $p_A(\lambda) = |\lambda I - A|$
  - Sufficient to consider controllability matrix: $\mathscr{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$

p_A **has degree** n
$\Rightarrow A^n$ **spanned by** $I, A, \ldots, A^{n-1}$

- Reachability: can we reach all states eventually?
  - If and only if $\operatorname{span}\mathscr{C} = \mathbb{R}^n \iff \operatorname{rank}\mathscr{C} = n$
- To reach x: control $\overrightarrow{u} = \mathscr{C}^{-1}(x - A^n x_0)$

Stabilizability

- Can we reach $x = 0$ eventually?

$$x_t = A^t x_0 + \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix} \begin{bmatrix} u_{t-1} \\ u_{t-2} \\ \vdots \\ u_0 \end{bmatrix}$$

- For each mode e_i (eigenvector of A):
  - Is $\|\lambda_i\| < 1$? $\Rightarrow$ stable, otherwise unstable
    - Stable modes reach 0 on their own
  - If unstable, is $e_i \in \operatorname{span}\mathscr{C}$? $\Rightarrow$ stabilizable, otherwise unstabilizable
    - Stabilizable modes = unstable, but controllable

- The system (A, B) is stabilizable if all modes are stable or stabilizable

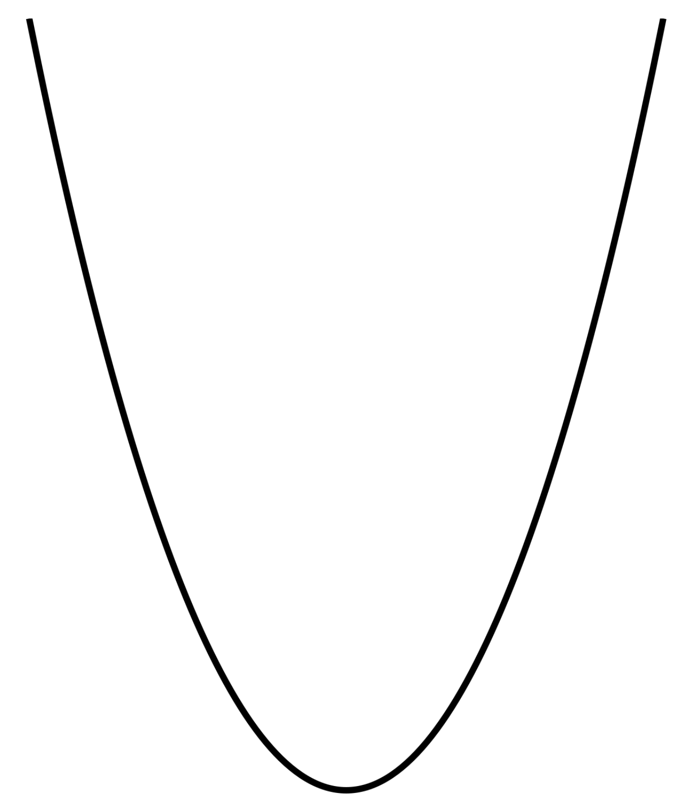
Today's lecture

Stability, reachability, stabilizability

Linear Quadratic Regulator

Hamiltonian

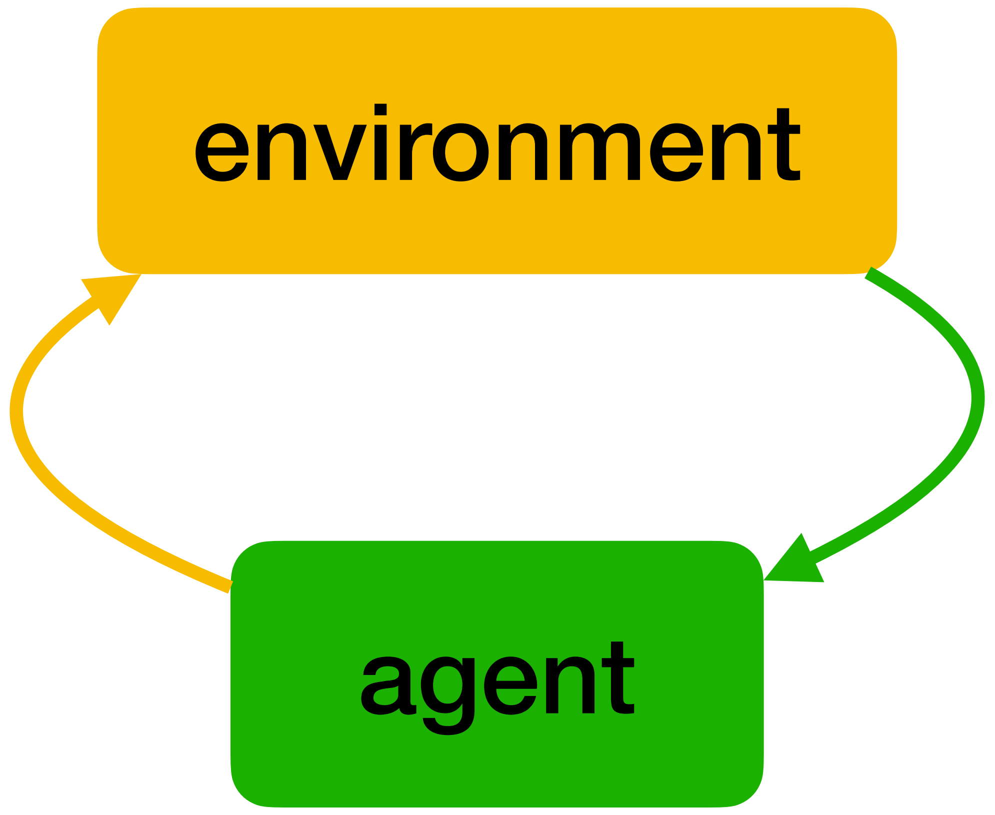
Quadratic costs



- Simplest reward: linear has no maximum ⇒ concave quadratic
  - Consider cost: $c(x_t, u_t) = \frac{1}{2}x_t^\intercal Q x_t + \frac{1}{2}u_t^\intercal R u_t$
- $Q \in \mathbb{R}^{n \times n}$ is positive semidefinite $Q \succeq 0$: $\frac{1}{2}x^\intercal Q x \geq 0$ for all x
  - No incentive to go to infinity in any direction
- $R \in \mathbb{R}^{m \times m}$ is positive definite $R \succ 0$: $\frac{1}{2}u^\intercal R u > 0$ for all u
  - Incentive for finite control in all directions
- Usually, finite or infinite horizon, no discounting

Linear Quadratic Regulator (LQR)

- Linear Quadratic Regulation (LQR) optimization problem:
  - Given LTI dynamics + quadratic cost (A, B, Q, R)
  - Find the control function $u_t = \pi(x_t)$
  - That minimizes $J^\pi = \sum_{t=0}^{T-1} c(x_t, u_t) = \frac{1}{2} \sum_{t=0}^{T-1} \left(x_t^\intercal Q x_t + u_t^\intercal R u_t \right)$
  - Such that $x_{t+1} = Ax_t + Bu_t$ for all t



Solving the LQR

- Bellman recursion: $V_t(x_t) = \min_{u_t} c(x_t, u_t) + V_{t+1}(x_{t+1})$

$x_{t+1} = Ax_t + Bu_t$

- Let's solve while also proving by induction that V_t is quadratic
  - Base case: $V_T \equiv 0$
  - Assume: $V_{t+1}(x_{t+1}) = \frac{1}{2} x_{t+1}^\intercal S_{t+1} x_{t+1} \qquad S_{t+1} \succeq 0$
  - Solve: $\nabla_{u_t}(c(x_t, u_t) + V_{t+1}(x_{t+1})) = 0$

Bellman optimality

$$0 = \nabla_{u_t}(c(x_t, u_t) + V_{t+1}(x_{t+1}))$$

$$V_{t+1}(x_{t+1}) = \tfrac{1}{2} x_{t+1}^\intercal S_{t+1} x_{t+1}$$
$$x_{t+1} = Ax_t + Bu_t$$

$$= \frac{1}{2}\nabla_{u_t}(x_t^\intercal Q x_t + u_t^\intercal R u_t + (Ax_t + Bu_t)^\intercal S_{t+1}(Ax_t + Bu_t))$$

$$= Ru_t + B^\intercal S_{t+1}(Ax_t + Bu_t)$$

$$u_t^* = -(R + B^\intercal S_{t+1} B)^{-1} B^\intercal S_{t+1} A x_t$$

- Plugging u_t^* into the Bellman recursion and rearranging terms:

$$V_t(x_t) = \frac{1}{2} x_t^\intercal (Q + A^\intercal (S_{t+1} - S_{t+1} B (R + B^\intercal S_{t+1} B)^{-1} B^\intercal S_{t+1}) A) x_t$$

- Ricatti equation: $S_t = Q + A^\intercal (S_{t+1} - S_{t+1} B (R + B^\intercal S_{t+1} B)^{-1} B^\intercal S_{t+1}) A$

Optimal control: properties

- Linear control policy: $u_t = L_t x_t$
  - Feedback gain: $L_t = -(R + B^\intercal S_{t+1} B)^{-1} B^\intercal S_{t+1} A$
- Quadratic value (cost-to-go) function $V_t(x_t) = \frac{1}{2} x_t^\intercal S_t x_t$
  - Cost Hessian $S_t = \nabla^2_{x_t} V_t$ is the same for all x_t
- Ricatti equation for S_t can be solved recursively backward

$$S_t = Q + A^\intercal (S_{t+1} - S_{t+1} B (R + B^\intercal S_{t+1} B)^{-1} B^\intercal S_{t+1}) A$$

  - Without knowing any actual states or controls (!) = at system design time

Infinite horizon

- Average cost: $J = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} c(x_t, u_t)$

- For each finite T we solve with Bellman recursion, affected by end $V_T \equiv 0$
  - In the limit, end effects go away ⇒ converge to time-independent

- Discrete-time algebraic Ricatti equation (DARE):

$$S = Q + A^\intercal(S - SB(R + B^\intercal SB)^{-1}B^\intercal S)A$$

- Optimal cost-to-go function: $V(x) = \frac{1}{2}x^\intercal Sx$; optimal cost: $J = \frac{1}{2}x_0^\intercal Sx_0$

Non-homogeneous case

- More generally, LQR can have lower-order terms

$$x_{t+1} = f_t(x_t, u_t) = A_t x_t + B_t u_t + b_t$$

$$c_t(x_t, u_t) = \frac{1}{2} x_t^\intercal Q_t x_t + \frac{1}{2} u_t^\intercal R_t u_t + u_t^\intercal N_t x_t + q_t^\intercal x_t + r_t^\intercal u_t + s_t$$

- More flexible modeling, e.g. tracking a target trajectory $\frac{1}{2}(x_t - \tilde{x}_t)^\intercal Q (x_t - \tilde{x}_t)$

- Solved essentially the same way
  - Cost-to-go $V_t(x_t)$ will also have lower-order terms

Today's lecture

Stability, reachability, stabilizability

Linear Quadratic Regulator

Hamiltonian

Co-state

$$c_t \in \mathbb{R} \qquad f_t \in \mathbb{R}^n$$

- Consider the cost-to-go $V_t^\pi(x_t) = c(x_t, u_t) + V_{t+1}^\pi(f(x_t, u_t))$
- To study its landscape over state space, consider its gradient

$$\nu_t = \nabla_{x_t} V_t^\pi = \nabla_{x_t} c_t + \nabla_{x_{t+1}} V_{t+1}^\pi \nabla_{x_t} f_t = \nabla_{x_t} c_t + \nu_{t+1} \nabla_{x_t} f_t$$

  - Jacobian of the dynamics: $\nabla_{x_t} f_t \in \mathbb{R}^{n \times n}$
- Co-state $\nu_t \in \mathbb{R}^n$ = direction of steepest increase in cost-to-go
  - Linear backward recursion $\nu_t = \nabla_{x_t} c_t + \nu_{t+1} \nabla_{x_t} f_t$; initialization: $\nu_T = 0$

Hamiltonian

- Cost-to-go recursion: (first-order approximation)

$$V_t^\pi(x_t) = c(x_t, u_t) + V_{t+1}^\pi(x_{t+1}) \approx c(x_t, u_t) + x_{t+1} \nabla_{x_{t+1}} V_{t+1}^\pi$$

co-state ν_{t+1}

- Hamiltonian = first-order approximation of the cost-to-go

$$\mathscr{H}_t(x_t, \nu_{t+1}, u_t) = c(x_t, u_t) + \nu_{t+1} f(x_t, u_t)$$

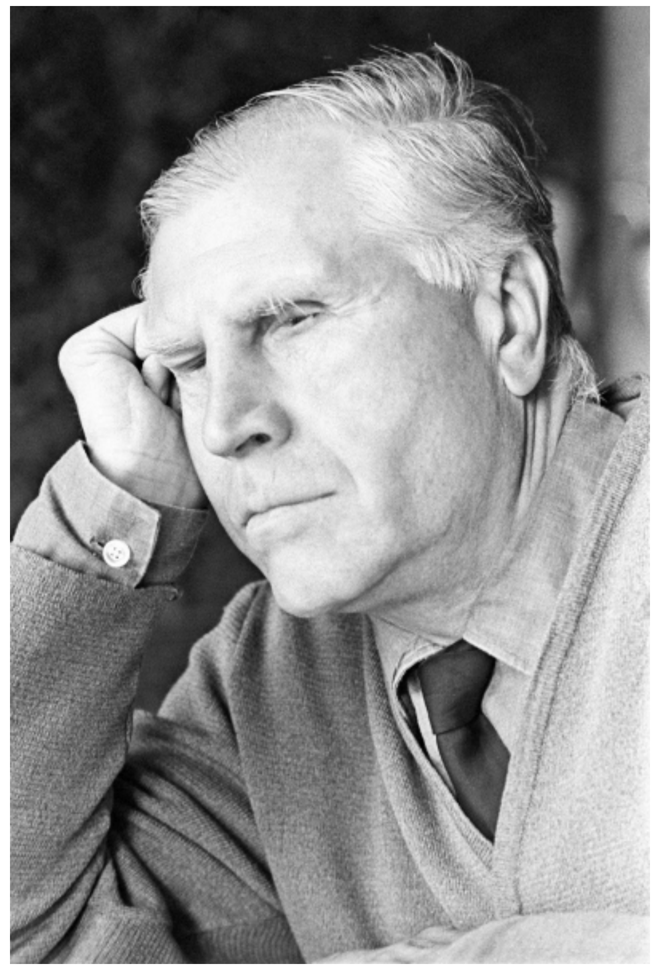
  - Related to, but not the same as the Hamiltonian in physics
- The Hamiltonian is useful to get first-order conditions for optimal control
  - Equivalent to Bellman optimality
  - Even more useful in continuous time (equivalent to Hamilton–Jacobi–Bellman)

Pontryagin's maximum principle

- Hamiltonian: $\mathscr{H}_t(x_t, \nu_{t+1}, u_t) = c(x_t, u_t) + \nu_{t+1} f(x_t, u_t)$

- Necessary optimality conditions:

$$\nabla_{x_t} \mathscr{H}_t = \nu_t \qquad \nabla_{\nu_{t+1}} \mathscr{H}_t = x_{t+1} \qquad \nabla_{u_t} \mathscr{H}_t = 0$$



Lev Pontryagin

- $\nabla_{x_t} \mathscr{H}_t = \nabla_{x_t} c_t + \nu_{t+1} \nabla_{x_t} f_t = \nu_t$ necessary for $\nu_t = \nabla_{x_t} V_t^\pi$ to be the co-state

- $\nabla_{\nu_{t+1}} \mathscr{H}_t = f(x_t, u_t) = x_{t+1}$ necessary for x_t to be the state for dynamics f

- Objective: $\min_\pi J$ s.t. $x_{t+1} = f(x_t, u_t)$; Lagrangian: $\mathscr{L} = \sum_{t=0}^{T-1} \mathscr{H}_t - \nu_{t+1} \cdot x_{t+1}$

Hamiltonian in LQR

- The Hamiltonian is generally high-degree, many local optima, hard to solve

- In LQR, the Hamiltonian is quadratic

$$\mathscr{H}_t = \frac{1}{2}x_t^\intercal Q x_t + \frac{1}{2}u_t^\intercal R u_t + \nu_{t+1}(Ax_t + Bu_t)$$

- This suggests forward–backward recursions for x, ν, and u:

$$x_{t+1} = \nabla_{\nu_{t+1}}\mathscr{H}_t = Ax_t + Bu_t$$

$$\nu_t = \nabla_{x_t}\mathscr{H}_t = \nu_{t+1}A + x_t^\intercal Q$$

$$\nabla_{u_t}\mathscr{H}_t = Ru_t + B^\intercal \nu_{t+1}^\intercal = 0$$

- The solution coincides with the Ricatti equations with $\nu_t^\intercal = S_t x_t \quad u_t = L_t x_t$

Recap

- LQR = simplest dynamics: linear; simplest cost: quadratic
- Can characterize stability, reachability, stabilizability in terms of (A, B)
- Can use Ricatti equation to find cost-to-go Hessian
- Equivalently: Hamiltonian gives state forward / co-state backward recursions