UCI University of California, Irvine

CS 277: Control and Reinforcement Learning

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Lecture 9: Stochastic Optimal Control

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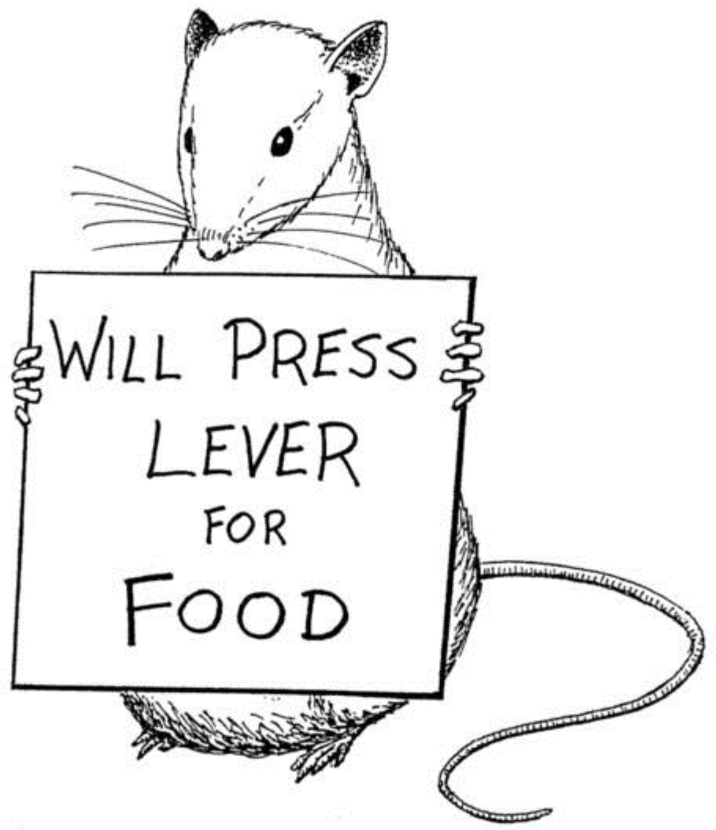


Logistics

assignments

- Assignment 2 due today

quizzes

- Quiz 4 will be published today
- Due Friday

Today's lecture

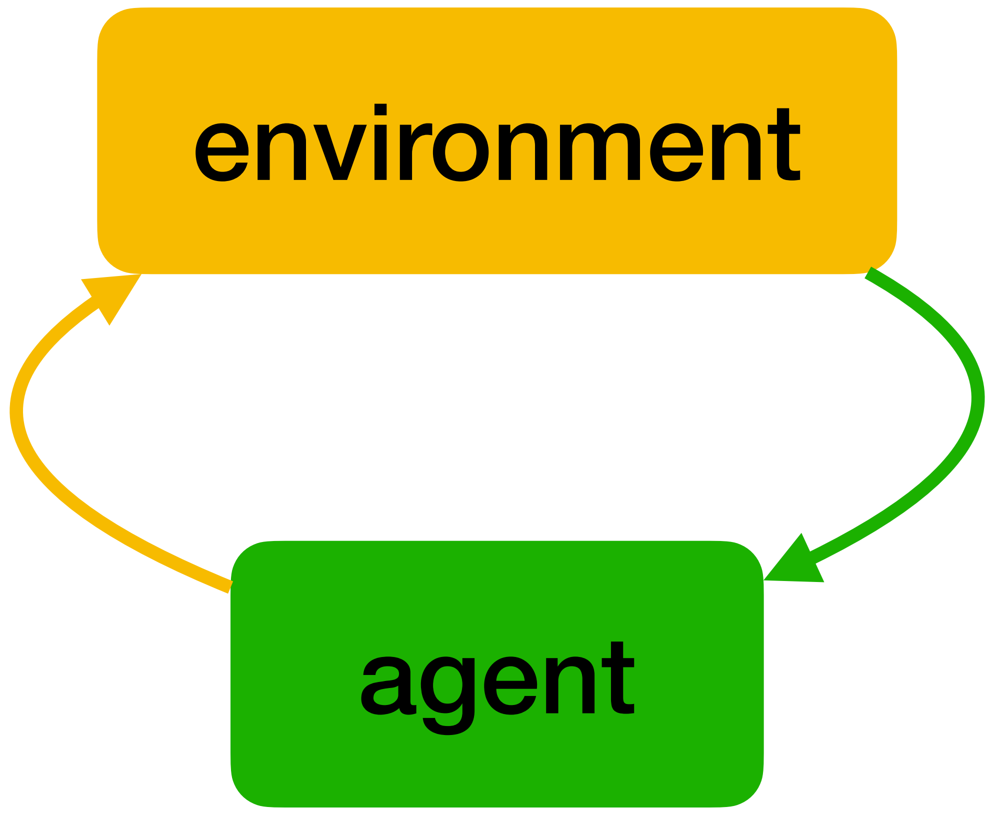
LQR with process noise

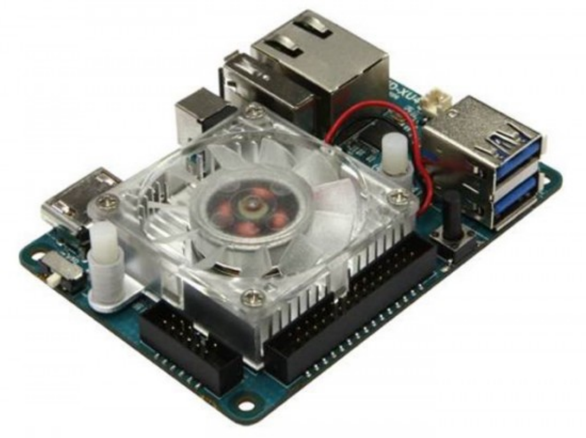
Linear–Quadratic Estimator

Linear–Quadratic–Gaussian control

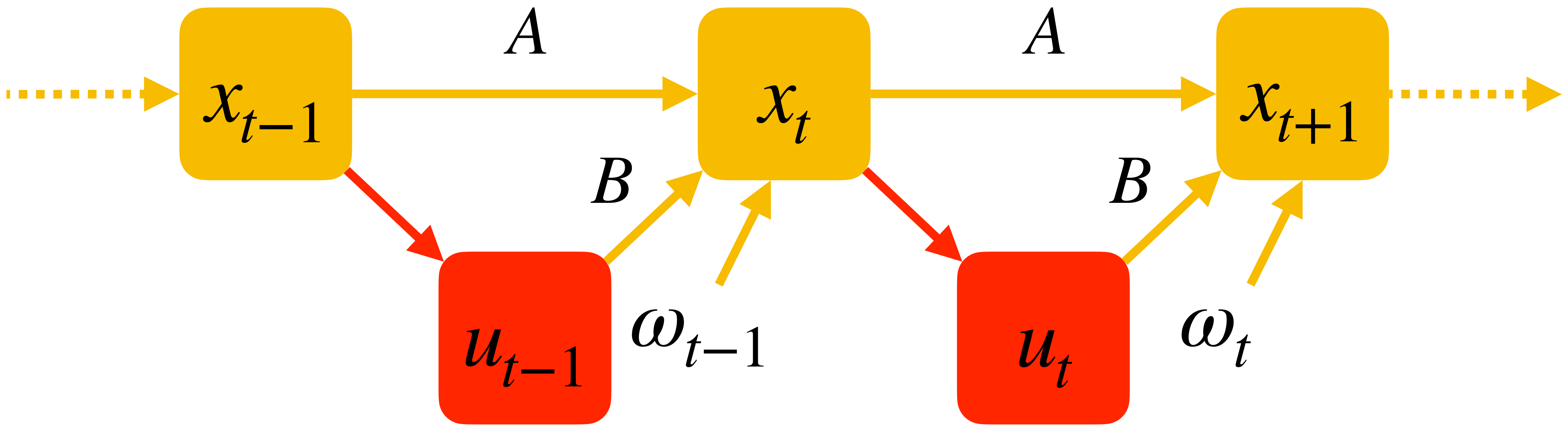
Reminder: Linear Quadratic Regulator (LQR)

- Linear Quadratic Regulation (LQR) optimization problem:
  - Given LTI dynamics + quadratic cost (A, B, Q, R)
  - Find the control function $u_t = \pi(x_t)$
  - That minimizes $J^\pi = \sum_{t=0}^{T-1} c(x_t, u_t) = \frac{1}{2} \sum_{t=0}^{T-1} \left(x_t^\intercal Q x_t + u_t^\intercal R u_t \right)$
  - Such that $x_{t+1} = Ax_t + Bu_t$ for all t





Stochastic control



- Simplest stochastic dynamics — Gaussian: $p(x_{t+1} | x_t, u_t) = \mathcal{N}(x_{t+1}; Ax_t + Bu_t, \Sigma_\omega)$

$$x_{t+1} = Ax_t + Bu_t + \omega_t \qquad \omega_t \sim \mathcal{N}(0, \Sigma_\omega) \qquad \Sigma_\omega \in \mathbb{R}^{n \times n}$$

  - Markov property: all ω_t are i.i.d for all t

- Why is there process noise?
  - Part of the state we don't model; Gaussian = maximum entropy given Σ_ω

- In continuous time = Langevin equation; Bu_t = external force

Stochastic optimal control

- Minimize expected cost-to-go

$$V_t^\pi(x_t) = \frac{1}{2}x_t^\intercal Q x_t + \frac{1}{2}u_t^\intercal R u_t + \mathbb{E}[V_{t+1}^\pi(x_{t+1}) \,|\, x_t, u_t = \pi(x_t)]$$

- Bellman equation:

$$V_t(x_t) = \min_{u_t} \frac{1}{2}x_t^\intercal Q x_t + \frac{1}{2}u_t^\intercal R u_t + \mathbb{E}_{(x_{t+1}|x_t,u_t)\sim\mathcal{N}(Ax_t+Bu_t,\Sigma_\omega)}[V_{t+1}(x_{t+1})]$$

- The cost-to-go is still quadratic, but with a free term
  - $x_t = 0$ is no longer absorbing $\Rightarrow V_t(0) \neq 0$

$$V_t(x_t) = \frac{1}{2}x_t^\intercal S_t x_t + V_t(0)$$

Solving the Bellman recursion

- Good to know: expectation of quadratic under Gaussian is $\mathbb{E}_{x\sim\mathcal{N}(\mu_x,\Sigma_x)}[x^\intercal Sx] = \mu_x^\intercal S\mu_x + \mathrm{tr}(S\Sigma_x)$

$$V_t(x_t) = \min_{u_t} \mathbb{E}_{(x_{t+1}|x_t,u_t)\sim\mathcal{N}(Ax_t+Bu_t,\Sigma_\omega)}\left[\frac{1}{2}x_t^\intercal Qx_t + \frac{1}{2}u_t^\intercal Ru_t + \underbrace{\frac{1}{2}x_{t+1}^\intercal S_{t+1}x_{t+1}} + V_{t+1}(0)\right]$$

$$= \min_{u_t}\left(\frac{1}{2}x_t^\intercal Qx_t + \frac{1}{2}u_t^\intercal Ru_t + \overbrace{\frac{1}{2}(Ax_t+Bu_t)^\intercal S_{t+1}(Ax_t+Bu_t) + \frac{1}{2}\mathrm{tr}(S_{t+1}\Sigma_\omega)} + V_{t+1}(0)\right)$$

new term, constant in u_t

- Linear control: $u_t^* = L_t x_t$ with same feedback gain: $L_t = -(R + B^\intercal S_{t+1}B)^{-1}B^\intercal S_{t+1}A$
- Same Ricatti equation for cost-to-go Hessian: $S_t = Q + A^\intercal(S_{t+1} - S_{t+1}B(R + B^\intercal S_{t+1}B)^{-1}B^\intercal S_{t+1})A$
- Cost-to-go: $V_t(x_t) = \frac{1}{2}x_t^\intercal S_t x_t + \sum_{t'=t+1}^{T}\frac{1}{2}\mathrm{tr}(S_{t'}\Sigma_\omega)$ — **noise–cost term, due to process noise**
  - Infinite horizon case: $\lim_{T\to\infty}\frac{1}{T}V_0(x_0) = \lim_{T\to\infty}\frac{1}{2T}\left(x_0^\intercal Sx_0 + \sum_{t=1}^{T}\mathrm{tr}(S\Sigma_\omega)\right) = \frac{1}{2}\mathrm{tr}(S\Sigma_\omega)$ — **state independent**

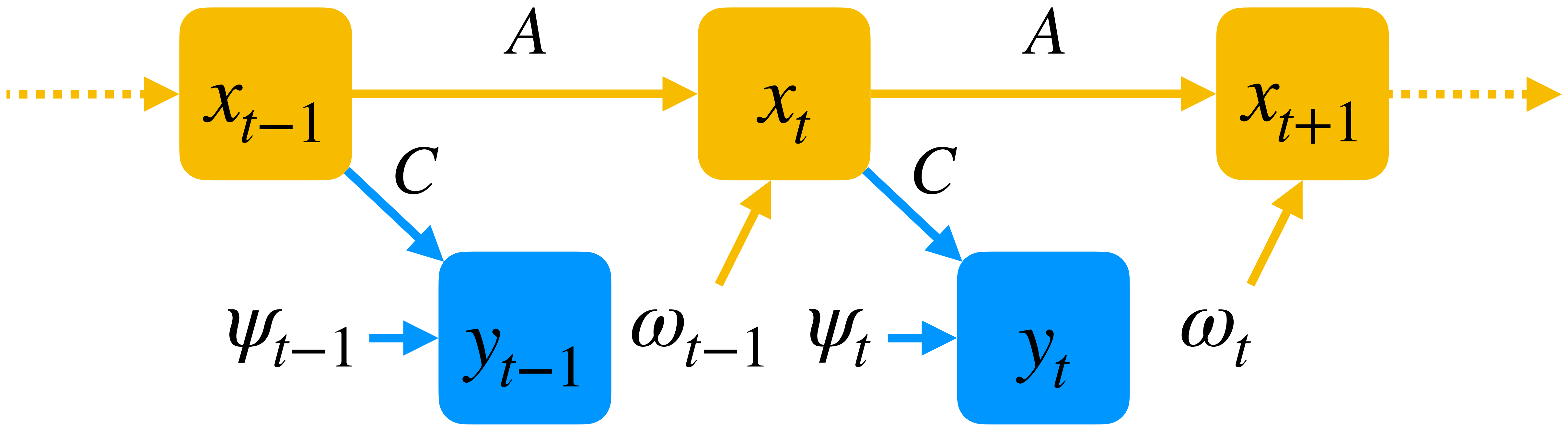
Today's lecture

LQR with process noise

Linear–Quadratic Estimator

Linear–Quadratic–Gaussian control

Partial observability



special case of Hidden Markov Model (HMM)

- What happens when we see just an observation $y_t \in \mathbb{R}^k$, not the full state x_t
  - Simplest observability model — Linear–Gaussian: $p(y_t | x_t) = \mathcal{N}(y_t; Cx_t, \Sigma_\psi)$

$$y_t = Cx_t + \psi_t \qquad \psi_t \sim \mathcal{N}(0, \Sigma_\psi) \qquad C \in \mathbb{R}^{k \times n}, \Sigma_\psi \in \mathbb{R}^{k \times k}$$

  - Markov property: all ω_t and ψ_t are independent, for all t
- Why is there observation noise?
  - Transient process noise that doesn't affect future states; only in agent's sensors

Gaussian Processes

- Jointly Gaussian variables: $\begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \Sigma_{(x,y)} = \begin{bmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_y \end{bmatrix} \right)$
  - Conditional distribution: $(x \mid y) \sim \mathcal{N}(\mu_{x|y}, \Sigma_{x|y})$

$$\mu_{x|y} = \mathbb{E}[x \mid y] = \mu_x + \Sigma_{xy}\Sigma_y^{-1}(y - \mu_y)$$

$$\Sigma_{x|y} = \mathrm{Cov}[x \mid y] = \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx} = \Sigma_{(x,y)}/\Sigma_y$$

Schur complement

  - Converse also true: if y and $(x \mid y)$ are Gaussian $\Longrightarrow$ (x, y) jointly Gaussian

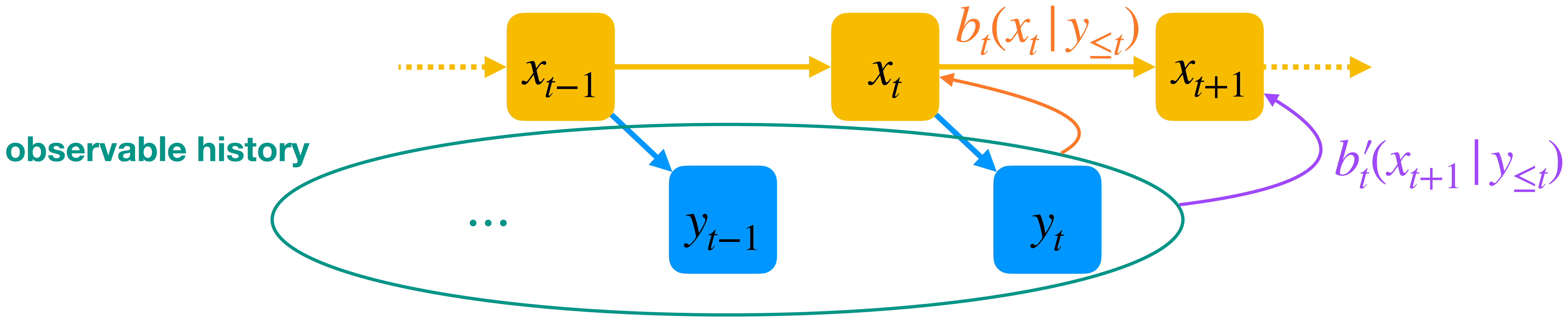
sufficient

- Gaussian Process (GP) $x_0, y_0, u_0, x_1, \ldots$: all variables are (pairwise) jointly Gaussian

Linear–Quadratic Estimator (LQE)

- Belief: our distribution over state x_t given what we know
- Belief given past observations (observable history): $b_t(x_t \mid y_{\le t})$
- b_t is sufficient statistic of $y_{\le t}$ for x_t = nothing more $y_{\le t}$ can tell us about x_t
  - In principle, we can update b_{t+1} only from b_t and y_{t+1} = filtering
  - Probabilistic Graphical Models terminology: belief propagation
- Linear–Quadratic Estimator (LQE): belief for our Gaussian Process
  - Update equations = Kalman filter

Belief and prediction



- Belief = what the observable history says of current state: $b_t(x_t \mid y_{\leq t})$
- Prediction = what the observable history says of next state: $b'_t(x_{t+1} \mid y_{\leq t})$
- In this Gaussian Process, both belief and prediction are Gaussian
  - Can be represented by their means $\hat{x}_t$, $\hat{x}'_{t+1}$ and covariances Σ_t, Σ'_{t+1}
  - Computed recursively forward

Kalman filter

- Given belief $b_t(x_t|y_{\le t}) = \mathcal{N}(\hat{x}_t, \Sigma_t)$, predict x_{t+1}:

$$\hat{x}'_{t+1} = \mathbb{E}[x_{t+1}|y_{\le t}] = \mathbb{E}[Ax_t + \omega_t|y_{\le t}] = A\hat{x}_t$$

$$\Sigma'_{t+1} = \mathrm{Cov}[x_{t+1}|y_{\le t}] = \mathrm{Cov}[Ax_t + \omega_t|y_{\le t}] = A\Sigma_t A^\intercal + \Sigma_\omega$$

- Given prediction $b'_t(x_t|y_{<t}) = \mathcal{N}(\hat{x}'_t, \Sigma'_t)$, update belief of x_t on seeing y_t:

$y_t = Cx_t +$ **noise** $\implies \Sigma_{x_t y_t|y_{<t}} = \Sigma_{x_t|y_{<t}} C^\intercal$

$$\hat{x}_t = \mathbb{E}[x_t|y_{\le t}] = \mu_{x_t|y_{<t}} + \Sigma_{x_t y_t|y_{<t}} \Sigma^{-1}_{y_t|y_{<t}} (y_t - \mu_{y_t|y_{<t}})$$

prediction error / innovation e_t

like conditioning x_t **on** y_t **and doing this given** $y_{<t}$

$$= \hat{x}'_t + \Sigma'_t C^\intercal (C\Sigma'_t C^\intercal + \Sigma_\psi)^{-1} \overbrace{(y_t - C\hat{x}'_t)}$$

$\Sigma_{y_t|y_{<t}} = C\Sigma_{x_t|y_{<t}} C^\intercal + \Sigma_\psi$

$$\Sigma_t = \mathrm{Cov}[x_t|y_{\le t}] = \Sigma_{x_t|y_{<t}} - \Sigma_{x_t y_t|y_{<t}} \Sigma^{-1}_{y_t|y_{<t}} \Sigma_{y_t x_t|y_{<t}}$$

$$= \Sigma'_t - \Sigma'_t C^\intercal (C\Sigma'_t C^\intercal + \Sigma_\psi)^{-1} C\Sigma'_t$$

Kalman filter

$e_t = y_t - C\hat{x}'_t$

- Linear belief update: $\hat{x}_t = A\hat{x}_{t-1} + K_t e_t = (I - K_t C)A\hat{x}_{t-1} + K_t y_t$

- Kalman gain: $K_t = \Sigma'_t C^\intercal (C\Sigma'_t C^\intercal + \Sigma_\psi)^{-1}$

- Covariance update — Ricatti equation:

$$\Sigma'_{t+1} = A(\Sigma'_t - \Sigma'_t C^\intercal (C\Sigma'_t C^\intercal + \Sigma_\psi)^{-1} C\Sigma'_t)A^\intercal + \Sigma_\omega$$

  - Compare to prior (no observations): $\Sigma_{x_{t+1}} = A\Sigma_{x_t} A^\intercal + \Sigma_\omega$

- Observations reduce covariance

  - Actual observation not needed to say by how much

Control as inference

- View Bayesian inference as optimization: minimizes MSE $\mathbb{E}[(x_t - \hat{x}_t)]$

- Control and inference are deeply connected:

$$\Sigma'_{t+1} = A(\Sigma'_t - \Sigma'_t C^\intercal (C \Sigma'_t C^\intercal + \Sigma_\psi)^{-1} C \Sigma'_t) A^\intercal + \Sigma_\omega$$

$$S_t = Q + A^\intercal (S_{t+1} - S_{t+1} B (R + B^\intercal S_{t+1} B)^{-1} B^\intercal S_{t+1}) A$$

- The shared form (Ricatti) suggests duality:

LQR	LQE
backward	forward
S_{T-t}	Σ'_t
A	$A^\intercal$
B	$C^\intercal$
Q	Σ_ω
R	Σ_ψ

- Information filter: recursion on $(\Sigma'_t)^{-1}$, presents a more principled duality

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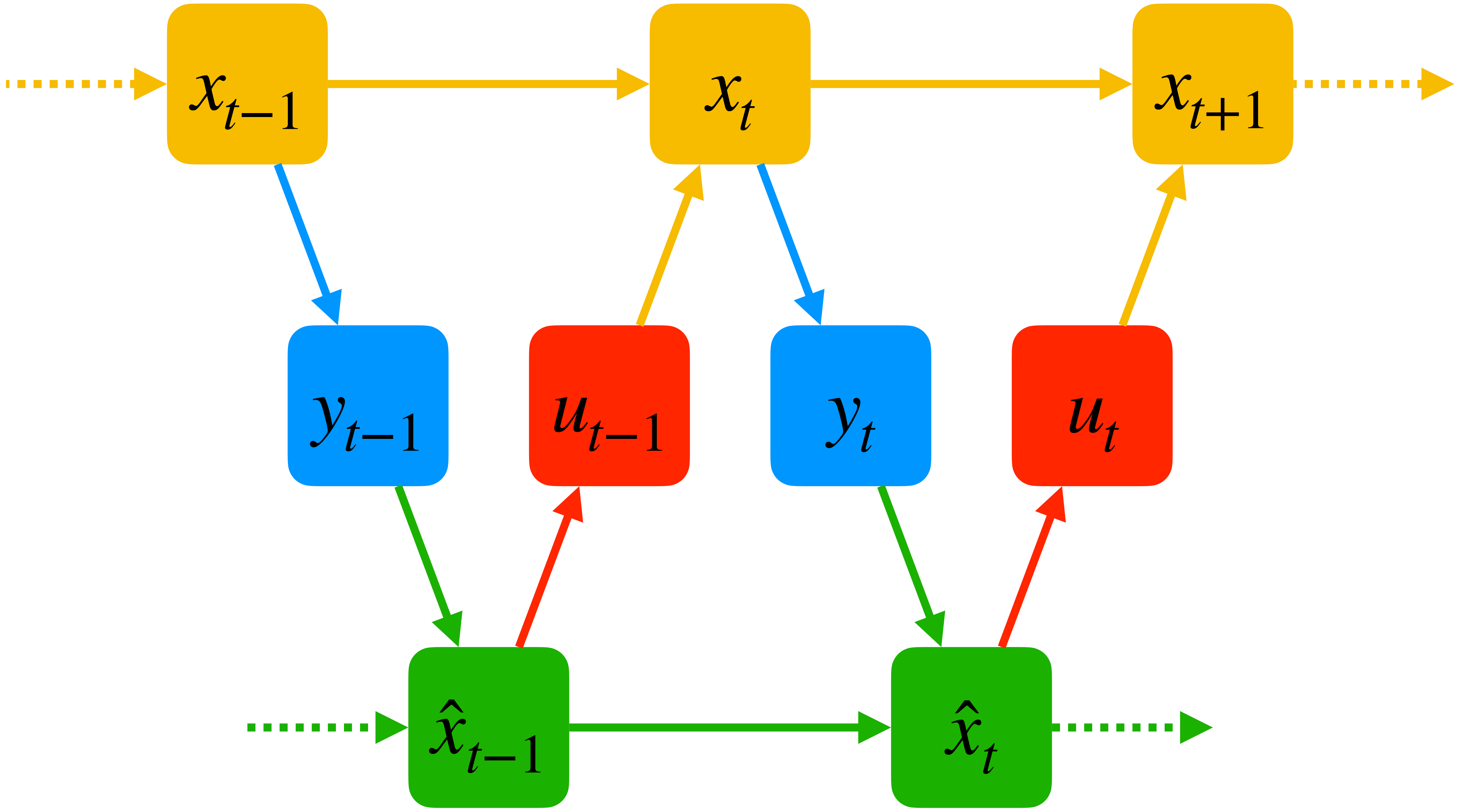
Linear–Quadratic–Gaussian control

Linear–Quadratic–Gaussian (LQG) control

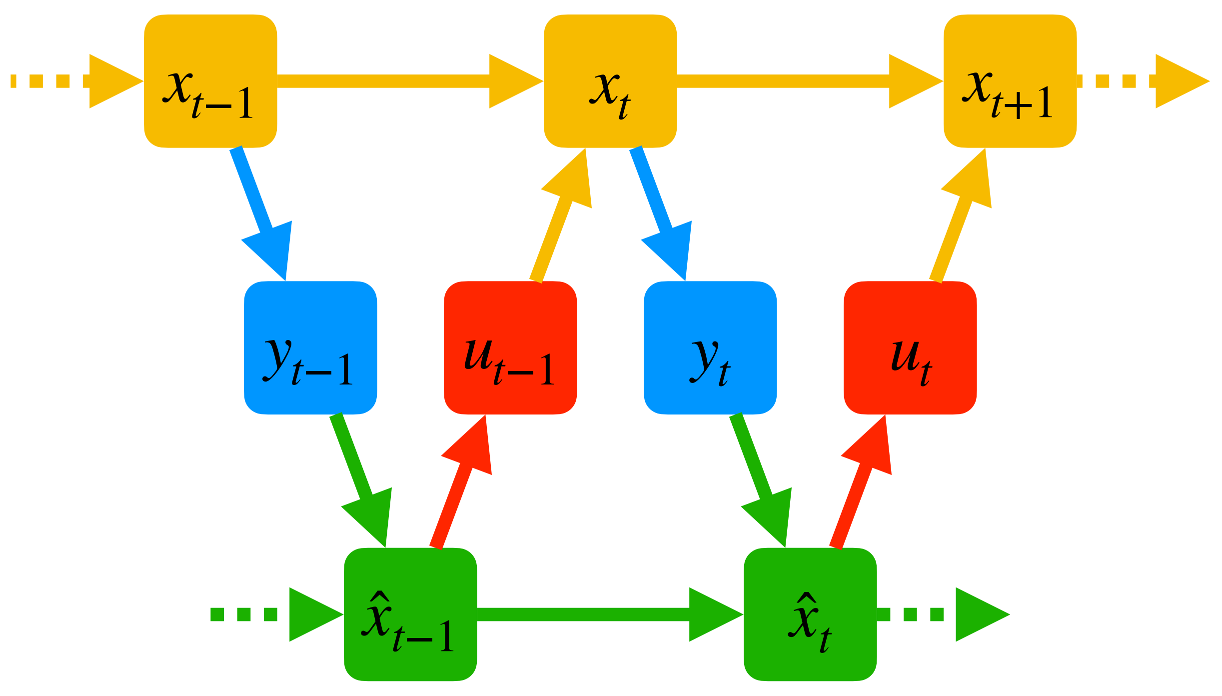


- Putting it all together:

$$x_{t+1} = Ax_t + Bu_t + \omega_t \qquad \omega_t \sim \mathcal{N}(0, \Sigma_\omega) \qquad \Sigma_\omega \in \mathbb{R}^{n \times n}$$

$$y_t = Cx_t + \psi_t \qquad \psi_t \sim \mathcal{N}(0, \Sigma_\psi) \qquad C \in \mathbb{R}^{k \times n}, \Sigma_\psi \in \mathbb{R}^{k \times k}$$

LQE with control



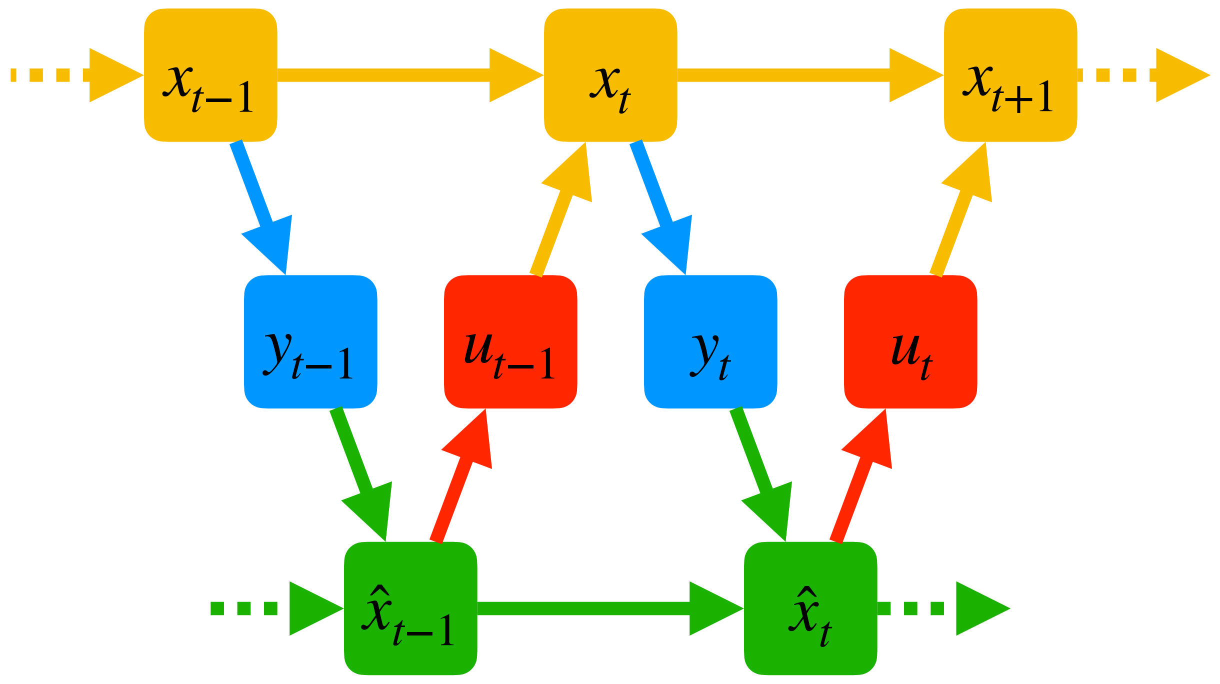
- How does control affect estimation?
  - Shifts predicted next state $\hat{x}'_{t+1} = A\hat{x}_t + Bu_t$
  - Bu_t known ⇒ no change in covariances ⇒ Ricatti equation still holds
  - Same Kalman gain K_t

$$\hat{x}_t = A\hat{x}_{t-1} + K_t e_t = (I - K_t C)(A\hat{x}_{t-1} + Bu_{t-1}) + K_t y_t$$

- And... that's it, everything else the same

LQR with partial observability

- Bellman recursion must be expressed in terms of what u_t can depend on: $\hat{x}_t$
  - Problem: but value depends on the true state x_t



- Value recursion for full state (environment + agent):

$$V_t^\pi(x_t, \hat{x}_t) = c(x_t, u_t) + \mathbb{E}[V_{t+1}^\pi(x_{t+1}, \hat{x}_{t+1}) \,|\, x_t, \hat{x}_t]$$

- In terms of only $\hat{x}_t$:

$\hat{x}_{t+1}$ **is sufficient for** x_{t+1}
⇒ separates it from $\hat{x}_t$

$$V_t^\pi(\hat{x}_t) = \mathbb{E}[V_t^\pi(x_t, \hat{x}_t) \,|\, \hat{x}_t] = \mathbb{E}[c(x_t, u_t) + V_{t+1}^\pi(x_{t+1}, \hat{x}_{t+1}) \,|\, \hat{x}_t] = \mathbb{E}[c(x_t, u_t) + V_{t+1}^\pi(\hat{x}_{t+1}) \,|\, \hat{x}_t]$$

- Certainty equivalent control: $u_t = L_t \hat{x}_t$ with the same feedback gain L_t

- And... that's it, everything else the same

LQG separability

Given $(A, B, C, \Sigma_\omega, \Sigma_\psi, Q, R)$, solve LQG = LQR + LQE separately

- LQR:
  - Compute value Hessian recursively backwards

$$S_t = Q + A^\intercal(S_{t+1} - S_{t+1}B(R + B^\intercal S_{t+1}B)^{-1}B^\intercal S_{t+1})A$$

  - Compute feedback gain:

$$L_t = -(R + B^\intercal S_{t+1}B)^{-1}B^\intercal S_{t+1}A$$

  - Control policy: $u_t = L_t\hat{x}_t$

- LQE:
  - Compute belief covariance recursively forward

$$\Sigma'_{t+1} = A(\Sigma'_t - \Sigma'_t C^\intercal(C\Sigma'_t C^\intercal + \Sigma_\psi)^{-1}C\Sigma'_t)A^\intercal + \Sigma_\omega$$

  - Compute Kalman gain:

$$K_t = \Sigma'_t C^\intercal(C\Sigma'_t C^\intercal + \Sigma_\psi)^{-1}$$

  - Belief update: $\hat{x}_t = A\hat{x}_{t-1} + K_t e_t$
  - with $e_t = y_t - C(A\hat{x}_{t-1} + Bu_{t-1})$

Recap

- Stochastic optimal control: control with process noise (stochastic dynamics)
  - Same concepts of controllability, but can't stop at $x_t = 0$
- LQE = linear–Gaussian observability $y_t = Cx_t + \psi_t$
  - Kalman filter = forward recursion to find belief $b_t(x_t \mid y_{\leq t})$
- LQG = LQR + LQE: estimate and control at the same time
  - Separability = optimal to solve LQR and LQE separately (only in LQG!)
  - Only differences: use $\hat{x}_t$ for control; add Bu_t to prediction